Subsidiary Maxima in Multiple Slit Interference: Algebraic Equations

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We present an analytical and numerical study on the subsidiary maxima in multiple slit interference with slit numbers between 12 and 31. By using the equation for the existing condition of the subsidiary maxima, we express the intensities at the maxima in a succinct analytical form. We also obtain algebraic equations for the phases for the subsidiary maxima with slit numbers between 12 and 31.

Keywords: Multiple slit, Subsidiary maxima, Diffraction

I. Introduction

Diffraction of laser light from multiple slits is a very important and fundamental phenomenon in general physics [1,2] and optics courses [3–7]. The intensity distribution pattern of diffracted light with a slit separation \(d\) and a negligible slit width is given by

\[
I = I_0 \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)},
\]

where \(I_0\) is the intensity due to single slit and \(N\) is the number of slits. In Eq. (1), \(\phi(= \frac{2\pi}{\lambda}d\sin\theta)\) is the phase difference between the lights emanating from the two adjacent slits, where \(\theta\) is the angular location of diffracted light and \(\lambda\) is the wavelength of light.

Typical intensity distribution patterns for \(N = 12\) and \(N = 13\) are shown in Fig. 1(a) and (b), respectively. In Fig. 1, the number of the subsidiary maxima between the two adjacent principal maxima is \(N - 2\). When \(N\) is even (odd), the phase \(\phi = \pi\) corresponds to the intensity minimum (subsidiary maximum). Because the subsidiary maxima are located at the symmetric positions relative to \(\phi = \pi\), when a subsidiary maximum is located at \(\phi\) (\(0 < \phi < \pi\)), the subsidiary maximum with the same intensity is located at \(2\pi - \phi\). The equation for the subsidiary maxima, obtained by differentiating Eq. (1) with respect to \(\phi\), is given by

\[
N \cot N\phi/2 = \cot \phi/2.
\]

We note that Eq. (2) also expresses the principal maxima and the central subsidiary maxima (when \(N\) is odd).

The analytical solutions of the subsidiary maxima for \(3 \leq N \leq 11\) were reported recently [8] and the behavior of the subsidiary maxima in the limiting region of \(N \to \infty\) was reported [9]. As reported in Ref. [8], it is not possible to obtain analytical solutions of the phases and amplitudes of the subsidiary maxima for \(N \geq 12\). This is because there are no analytical solutions for the algebraic equations higher than quartic. However, we can obtain algebraic equations even in these cases, which are more amenable than Eq. (2). Extending the previous studies [8,9], in this paper we present algebraic equations for the subsidiary maxima in multiple slit interference with \(12 \leq N \leq 31\). We also present the intensities at the subsidiary maxima as much simpler forms than those presented in Ref. [8].

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II. Intensity at the subsidiary maxima

In the previous report, we presented the analytical form of the intensities at the subsidiary maxima in a polynomial of the phases for the subsidiary maxima [8]. However, it is possible to obtain the intensities as a much simpler form. From the equation for the maxima in Eq. (2), we obtain the following:

$$\sin^2 \frac{N \phi}{2} = \frac{2N^2 \sin^2 \frac{\phi}{2}}{N^2 + 1 + \cos \phi (1 - N^2)}.$$  

(3)

Inserting Eq. (3) into Eq. (1), the intensity at the maxima is given by

$$I = I_0 \frac{2N^2}{N^2 + 1 + (1 - N^2) \cos \phi}.$$  

(4)

Equation (4) is valid even for the cases of the principal maxima and central subsidiary maxima (when \(N\) is odd). For the principal maxima, since \(\cos \phi = 1\), we have \(I = N^2 I_0\). In the case of central subsidiary maxima, we have \(I = I_0\) because \(\cos \phi = -1\).

Because the intensities at the subsidiary maxima for \(3 \leq N \leq 7\) are explicitly given in Ref. [8], we express the intensities only for \(8 \leq N \leq 11\) using Eq. (4) more concisely. When \(N = 8\), the intensities at the six subsidiary maxima are given by [8]

$$I_i = \frac{128}{16 \, 807} (162z_i^2 + 306z_i + 281) I_0,$$

for \(i = 1, 2, 3, 4, 5, 6\) \(= I_1, I_2, I_3, I_4, I_5, I_6\),

(5)

where the analytical expressions for \(z_i(= \cos \phi_i)\) with \(i = 1, 2, \ldots, 6\) are presented in Ref. [8]. Instead of Eq. (5), the intensities are given by simply using Eq. (4)

$$I_i = \frac{128}{65 - 63z_i} I_0,$$

for \(i = 1, 2, \ldots, 6\).

(6)

We can express the intensities in Eqs. (45), (48), and (51) in Ref. [8] as follows:

$$I_i = \frac{81}{41 - 40z_i} I_0,$$

for \(N = 9\),

(7)

$$I_i = \frac{200}{101 - 99z_i} I_0,$$

for \(N = 10\),

(8)

$$I_i = \frac{121}{61 - 60z_i} I_0,$$

for \(N = 11\),

(9)

respectively, with \(i = 1, 2, \ldots, N - 2\). The explicit values of \(z_i\) are given in Ref. [8].

III. Results for \(12 \leq N \leq 31\)

We consider the number of slits \(N = 12\). Then, Eq. (2) can be written as follows:

$$12 \cot 6 \phi = \cot \frac{\phi}{2},$$  

(10)

which becomes subsequently

$$12 \cot 12 \alpha = \cot \alpha,$$  

(11)

where \(\alpha \equiv \phi/2\). We expand Eq. (11) using a trigonometric expansion formula, and Eq. (11) can be expressed as:

$$x \left(5632x^{10} - 16640x^8 + 18304x^6 - 9152x^4 + 2002x^2 - 143\right) = 0,$$

(12)

where \(x \equiv \sin \alpha = \sin(\phi/2)\). In Eq. (12), the solution for \(x = 0\) implies \(\phi = 0, 2\pi, 4\pi, \ldots\), that correspond to the principal maxima. Thus, the equation for the subsidiary maxima is derived from Eq. (12) as follows:

$$176z^5 + 160z^4 - 112z^3 - 96z^2 + 9z + 6 = 0,$$

(13)

where we use the identity \(x^2 = (1 - z)/2\) with \(z \equiv \cos \phi\). Unlike the cases in Ref. [8], because Eq. (13) is a quintic algebraic equation that does not have analytical solutions, we have to solve Eq. (13) numerically. Then, the results are given by

$$\phi_1 = 0.750660, \phi_2 = 1.29062, \phi_3 = 1.82183,$$

$$\phi_4 = 2.3504, \phi_5 = 2.87795,$$

$$\phi_6 = 2\pi - \phi_5, \phi_7 = 2\pi - \phi_4, \phi_8 = 2\pi - \phi_3,$$

$$\phi_9 = 2\pi - \phi_2, \phi_{10} = 2\pi - \phi_1,$$

(14)

and the intensities in Eq. (4) for \(N = 12\) are given by

$$I_i = \frac{288}{145 - 143 \cos \phi_i^6} I_0,$$

for \(i = 1, 2, \ldots, 10\).

(15)

From Eqs. (14) and (15), the intensities are explicitly given by

$$I_1 = 7.12292I_0, \quad I_2 = 2.73096I_0, \quad I_3 = 1.59537I_0,$$

$$I_4 = 1.17298I_0, \quad I_5 = 1.01746I_0,$$

$$I_6 = I_5, \quad I_7 = I_4, \quad I_8 = I_3,$$

$$I_9 = I_2, \quad I_{10} = I_1.$$  

(16)
Table 1. The coefficients $a_j$ ($j = 0, 1, \cdots, \left[ \frac{N}{2} \right] - 1$) for the slits with $14 \leq N \leq 31$.

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We calculated the phases and intensities for $N = 13$ in a similar method. In this case, Eq. (2) becomes the following equation:

$$13 \cot \frac{13\phi}{2} = \cot \frac{\phi}{2},$$

(17)

that can be transformed to the following equation:

$$x \sqrt{1 - x^2} (6144x^{10} - 16640x^8$$

$$+ 16640x^6 - 7488x^4 + 1456x^2 - 91) = 0.$$  (18)

In Eq. (18), the solution $x = 0$ implies again the conditions for the principal maxima as in the case for $N = 12$. In contrast, the solutions for $\sqrt{1 - x^2} = 0$, i.e., $\phi = \pi, 3\pi, 5\pi, \cdots$ correspond to the central subsidiary maxima, which exist only for an odd $N$. Thus, the algebraic equation for the phases for the subsidiary maxima other than central subsidiary maxima is given by

$$192z^5 + 80z^4 - 160z^3 - 48z^2 + 24z + 3 = 0.$$  (19)

Then, the explicit results of the phases are given by

$$\phi_1 = 0.692674, \quad \phi_2 = 1.19091, \quad \phi_3 = 1.68105$$

$$\phi_4 = 2.16869, \quad \phi_5 = 2.65534, \quad \phi_6 = \pi,$$

$$\phi_7 = 2\pi - \phi_5, \quad \phi_8 = 2\pi - \phi_4, \quad \phi_9 = 2\pi - \phi_3$$

$$\phi_{10} = 2\pi - \phi_2, \quad \phi_{11} = 2\pi - \phi_1.$$  (20)

The intensities for $N = 13$ are given by

$$I_i = \frac{169}{85 - 84\cos \phi_i} I_0, \quad \text{for } i = 1, 2, \cdots, 11,$$

(21)

and are given by explicitly

$$I_1 = 8.30119I_0, \quad I_2 = 3.13825I_0, \quad I_3 = 1.79325I_0$$

$$I_4 = 1.27755I_0, \quad I_5 = 1.06114I_0, \quad I_6 = I_0,$$

$$I_7 = I_5, \quad I_8 = I_1, \quad I_9 = I_4$$

$$I_{10} = I_2, \quad I_{11} = I_1.$$  (22)

In general, the algebraic equation for the phases with a slit number $N$ is given by

$$\sum_{j=0}^{[\frac{N}{2}] - 1} a_j z^j = 0,$$

(23)

where $[y]$ is the largest integer not greater than $y$. The phases are given by $\phi_i = \cos^{-1} z_i$ where $z_i$ is the $i_{th}$ solution of Eq. (23), and the intensity is given in Eq. (4). For the slits with $14 \leq N \leq 31$, the coefficients $a_i$ are presented in Table 1.

IV. Conclusion

In this paper we presented analytical and numerical studies on the subsidiary maxima in multiple slit interference with negligible slit width. We obtained the intensity at the subsidiary maxima in a succinct form as in Eq. (4). Because it was not possible to obtain analytical solutions for the slits with $N \geq 12$, we obtained algebraic equations for the phase for the slits with $12 \leq N \leq 31$. The obtained results would be useful in fast prediction of the phase and the intensity for the subsidiary maxima.
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