Calculation of the off-axes Magnetic Field for Finite-length Solenoids

Taehun JANG
Department of Physics, Kyungpook National University, Daegu 41566, Korea

Yun Kyung SEO · Sang Ho SOHN*
Department of Science Education, Graduate School, Kyungpook National University, Daegu 41566, Korea

JoongWoo JUNG
Haknam High School, Daegu 41420, Korea

(Received 12 June 2020 : accepted 29 June 2020)

In this study, we derived an approximate analytic function for the off-axis magnetic field of a finite-length solenoid by using the magnetic vector potential of a circular current loop. We verified that the derived analytic function reduced to a well-known magnetic field formula on the vertical axis of the solenoid and also inferred the magnetic field on the horizontal axis of the solenoid. Furthermore, we investigated the magnetic field at arbitrary points satisfying the approximate conditions through a simulation performed using Wolfram Mathematica.

Keywords: Magnetic field, Off-axes solenoid, Analytic function, Mathematical simulation

I. Introduction

A solenoid is a coil that is wound many times along an axis, and a magnetic field is generated around the solenoid when a current flows through it. When a large current flows through a solenoid, a strong magnetic field is generated inside it. Therefore, solenoids are used as strong magnets. Furthermore, because of magnetic forces acting on the ends of a solenoid, it is also used as an electrically controlled valve. Although solenoid is employed in commonly used devices such as microphones, speakers, transformers, and metal detectors, students learn about solenoids in high school. In high school curriculums, an exploration activity involving the observation of how a compass needle changes direction around a solenoid as the current flowing through the solenoid changes is introduced, and an experiment in which the magnetic field of a solenoid is measured as a function of currents is included [1]. In university textbooks, a refined formula for the magnetic field on the central axis of a finite-length solenoid is derived. In particular, the magnetic field on the central axis is uniform when the solenoid is infinitely long [2].

A recent study considered a magnet to be a solenoid and theoretically calculated the magnetic force between the magnet and a solenoid on the basis of the mutual inductance effect [3]. While magnetic fields produced by solenoids have been studied in previous works [4–6], most of the studies investigated magnetic fields on the vertical central axis, namely z-axis of the cylindrical coordinate. Knowledge of the magnetic field at a point away from a solenoid axis is useful for determining the electromotive force induced on the solenoid by a magnet moving around the solenoid as well as the magnetic force between the magnet and the solenoid. Accordingly, theoretical formulas for magnetic fields on the off-axis of a solenoid are required and several previous studies have used the Biot-Savart law [7], magnetic scalar potential [8], or magnetic vector potential [9] to determine the off-axis magnetic field. However, because the formulas...
obtained by these studies contain elliptic integrals, the analytic function is not available.

In this study, an approximate analytic function was derived for the magnetic field at arbitrary points of an off-axis of solenoid. In addition, the magnetic fields at points within an approximate limit around a solenoid were obtained via a simulation using Wolfram Mathematica, and they are discussed here. The analytic functions derived in the present study for off-axis magnetic fields for a solenoid can be useful to determine the magnetic field near the central axis of electromagnets and far the surface of permanent magnets, which are considered as solenoids, as well as the magnetic force on a magnet moving outside a solenoid.

II. Theoretical calculations

In this study, the following theoretical approach was used for calculating the magnetic field at an arbitrary point of off-axis of solenoid. First, we considered the solenoid as a collection of circular current loops and calculated the vector potentials $A_\phi(r, \theta)$ and $A_\phi(\rho, z)$ at arbitrary positions $(r, \theta)$ and $(\rho, z)$ for a single circular current loop. The elliptic integrals appearing in the equations were represented as power series by considering appropriate approximations to obtain the analytic function for $A_\phi$. Second, we obtained the magnetic field of the solenoid by first determining the magnetic field of a single circular current loop from the relation $\mathbf{B} = \nabla \times \mathbf{A}$ and then integrating it along the z-axis.

The magnetic field formula derived was validated by comparing the magnetic fields on the central z-axis obtained from the formula with those obtained for the z-axis from a well-known magnetic field formula. Magnetic fields at points on the horizontal axis at $z = 0$ were also determined from the derived formula. Finally, a simulation was performed using Wolfram Mathematica to calculate magnetic fields at arbitrary points around the solenoid.

1. Magnetic fields $\mathbf{B}(r, \theta)$ and $\mathbf{B}(\rho, z)$ at arbitrary points $(r, \theta)$ and $(\rho, z)$ around a circular current loop

Figure 1 shows the magnetic field $\mathbf{B}(r, \theta)$ at arbitrary point $(r, \theta)$ on a circular current loop.

In the Fig. 1, the vector potential $\mathbf{A}$ can be defined as

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l'}}{r}$$

Furthermore, $r'$ and $d\mathbf{l'}$ are given by

$$r' = \left( r^2 + \alpha^2 - 2ar \sin \theta \cos \phi' \right)^{1/2}$$
$$d\mathbf{l'} = a(-\sin \phi' \hat{x} + \cos \phi' \hat{y}) d\phi'$$

and the vector potential $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \frac{\mu_0 Ia}{4\pi} \int_0^{2\pi} \frac{(-\sin \phi' \hat{x} + \cos \phi' \hat{y})}{(a^2 + r^2 - 2ar \sin \theta \cos \phi')^{1/2}} d\phi'$$

The azimuthal integration path of the magnetic field at the periphery of the circular current loop is symmetric about $\phi' = 0$. Therefore, the $x$-component disappears and only the $y$-component of $A_\phi$ remain [10].

$$A_\phi(r, \theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos \phi'}{(a^2 + r^2 - 2ar \sin \theta \cos \phi')^{1/2}} d\phi'$$

Substituting $\phi' = 2\varphi + \pi$ in this equation gives $d\phi' = 2d\varphi$ and $\varphi = -\frac{\pi}{2} \sim +\frac{\pi}{2}$, and the use of the relation

$$k^2 = \frac{4a \sin \theta}{a^2 + r^2 + 2ar \sin \theta}$$

yields

$$\left(a^2 + r^2 - 2ar \sin \theta \cos \phi' \right)^{1/2} = \left( a^2 + r^2 + 2ar \sin \theta \right)^{1/2} (1 - k^2 \sin^2 \phi')^{1/2}$$

Fig. 1. (Color online) Magnetic field $\mathbf{B}(r, \theta)$ at arbitrary points $(r, \theta)$ on a circular current loop.

New Physics: Sae Mulli, Vol. 70, No. 8, August 2020
and
\[ \cos \varphi' = \cos (2 \cdot \varphi' = 1 - 2 \sin^2 \varphi' = 1 - 2 \sin^2 (\varphi + \pi) = 1 - 2 \cos^2 \varphi = 1 - 2(1 - \sin^2 \varphi) = 2sin^2 \varphi - 1 \] (5)

Therefore, equation (4) can be expressed as
\[ A_\varphi (r, \theta) = \frac{\mu_0 I a}{4\pi} \frac{4}{(a^2 + r^2 + 2ar \sin \theta)^{1/2}} \times \int_0^{\pi/2} \frac{2sin^2 \varphi - 1}{(1 - k^2sin^2 \varphi)^{1/2}} d\varphi \] (6)

Furthermore, we can use the mathematical relation
\[ \sin^2 \varphi = \frac{1 - (1 - k^2sin^2 \varphi)}{k^2} \]

to modify equation (6) as
\[ \int_0^{\pi/2} \frac{2sin^2 \varphi - 1}{(1 - k^2sin^2 \varphi)^{1/2}} d\varphi = \int_0^{\pi/2} \frac{2}{(1 - k^2sin^2 \varphi)^{1/2}} d\varphi = \frac{2}{k^2} \] (7)

where
\[ K(k^2) = \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2sin^2 \varphi)^{1/2}} \]
(8)

\[ E(k^2) = \int_0^{\pi/2} (1 - k^2sin^2 \varphi)^{1/2} d\varphi \]
(9)

Equations (8) and (9) are the complete elliptic integrals of the first and second kind, respectively. Vector potential
\[ A_\varphi (r, \theta) = \frac{\mu_0 I a}{4\pi} \frac{4}{(a^2 + r^2 + 2ar \sin \theta)^{1/2}} \times \left[ \left( 2 - k^2 \right) K(k^2) - 2E(k^2) \right] \] (10)

Equation (10) is a formula for calculating the magnetic field at arbitrary point \((r, \theta)\) of a circular current loop, but it is difficult to use because the elliptic integral is not in a closed form (i.e., it is not in the form of an analytic function). Moreover, in the case of solenoids, we must integrate equation (10) in the z-axis direction, in which case even the elliptic integration of the third kind should be considered [9].

In this study, the aim was to obtain the magnetic field at arbitrary points around a solenoid. In general, the magnetic field inside an infinitely long solenoid is almost similar to that on the central axis, and it is known to be very weak near the outside of the solenoid [11]. For a finite-length solenoid, it is possible to expand the elliptic integrals in equation (10) under the conditions \(r \gg a, a \gg r\), or \(\sin \theta \approx 0\) as [12]
\[ K(k^2) = 1 + k^2 + \frac{9k^4}{64} + \cdots \] (11)

and
\[ E(k^2) = 1 - k^2 + \frac{3k^4}{64} + \cdots \] (12)

When only the term \(k^2\) considered, the value of the expression in the square brackets of Equation (10) is \(k^2/16\). Therefore, the vector potential can be expressed as
\[ A_\varphi (r, \theta) = \frac{\mu_0 I a}{4} \frac{4}{(a^2 + r^2 + 2ar \sin \theta)^{3/2}} \] (13)

In cylindrical coordinates, equation (13) can be transformed as
\[ A_\varphi (\rho, z) = \frac{\mu_0 I a^2}{4} \frac{\rho^2}{(\rho^2 + z^2 + a^2 + 2a\rho)^{3/2}} \] (14)

From the relation \(\mathbf{B} = \nabla \times \mathbf{A}\), the magnetic field components \(B_r, B_\theta, B_\varphi\) of the magnetic field \(\mathbf{B}\) in spherical coordinates can be expressed as
\[ B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\varphi (r, \theta)) = \frac{\mu_0 I a^2}{4} \frac{\sin \theta}{(a^2 + r^2 + 2ar \sin \theta)^{5/2}} \] (15)

\[ B_\theta = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (r A_\varphi (r, \theta)) = \frac{\mu_0 I a^2}{4} \frac{\sin \theta}{(r^2 - 2a^2 - ar \sin \theta)^{5/2}} \] (16)

\[ B_\varphi = 0 \] (17)

In cylindrical coordinate, the magnetic field components can be expressed as
\[ B_\rho = -\frac{\partial A_\varphi (\rho, z)}{\partial z} = \frac{3\mu_0 I a^2 \rho z}{4(\rho^2 + z^2 + a^2 + 2a\rho)^{5/2}} \] (18)

\[ B_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\varphi (\rho, z)) = \frac{\mu_0 I a^2}{4} \frac{2(\rho^2 + z^2 + a^2 + 2a\rho) - 3\rho(\rho + a)}{(\rho^2 + z^2 + a^2 + 2a\rho)^{5/2}} \] (19)
If the solenoid is considered as a combination of circular current loops, \( \mathbf{B}(r, \theta) \) and \( \mathbf{B}(\rho, z) \) can be derived from the relation \( \mathbf{B} = \nabla \times \mathbf{A} \) after integrating equations (13) and (14) along the \( z \)-axis. However, because integrating equations (15), (16), (18), and (19) along the \( z \)-axis is more convenient for the calculation of magnetic field, we adopted this approach in this study.

2. Magnetic fields \( \mathbf{B}(r, \theta) \) and \( \mathbf{B}(\rho, z) \) at arbitrary points \( (r, \theta) \) and \( (\rho, z) \) around a solenoid

Figure 2 shows the magnetic field \( \mathbf{B}(r, \theta) \) at arbitrary point \( (r, \theta) \) around a solenoid with \( N \) turns. \( l \) denotes the distance from the center of the solenoid to the field source. Also we define the vertical axis as \( z \)-axis of a cylindrical coordinate while the horizontal axis as \( \rho \)-axis.

Substituting \( r + \Delta = r + l \cos \theta \) and \( \frac{\pi}{2} Idl(= nIdl) \) equations (15) and (16) gives

\[
d B_r = \frac{\mu_0 n I a^2 \sin \theta}{4} \times \frac{[2(r + l \cos \theta)^2 + a(r + l \cos \theta) \sin \theta + 2a^2]}{[(r + l \cos \theta)^2 + 2a(r + l \cos \theta) \sin \theta + a^2]^{5/2}} dl
\]

(20)

By replacing \( r + l \cos \theta \equiv t \) and integrating equations (20) and (21) from \(-L/2\) to \(L/2\), we can derive

\[
B_r(r, \theta) = \frac{\mu_0 n I a^2}{4} [P + Q + R]
\]

(22)

where

\[
P = \frac{2p^3}{3(2pa \sin \theta + a^2)(p^2 + 2pa \sin \theta + a^2)^{3/2}}
\]

\[
Q = \frac{a \sin \theta}{3(q^2 + 2qa \sin \theta + a^2)^{3/2}} - \frac{a \sin \theta}{3(p^2 + 2pa \sin \theta + a^2)^{3/2}}
\]

\[
R = \frac{2pa^2(2p^2 + 6pa \sin \theta + 3a^2)}{3(2pa \sin \theta + a^2)^2(2p^2 + 2pa \sin \theta + a^2)^{3/2}} - \frac{2qa^2(2q^2 + 6qa \sin \theta + 3a^2)}{3(2qa \sin \theta + a^2)^2(2q^2 + 2qa \sin \theta + a^2)^{3/2}}
\]

(23)

(24)

(25)

Here, \( p \) and \( q \) are given by \( r + l/2 \cos \theta \) and \( r - l/2 \cos \theta \), respectively.

For the vertical central axis \( (r = z, \theta = 0) \) of the solenoid, equation (22) reduces to

\[
B_z(z, 0) = \frac{\mu_0 n I a^2}{4} \left[ (P)_{r=z, \theta=0} + (Q)_{r=z, \theta=0} + (R)_{r=z, \theta=0} \right]
\]

\[
= \frac{\mu_0 n I}{2} \left[ \frac{z + \frac{l}{2}}{((z + \frac{l}{2})^2 + a^2)^{1/2}} - \frac{z - \frac{l}{2}}{((z - \frac{l}{2})^2 + a^2)^{1/2}} \right]
\]

(26)

which is a well-known formula for the magnetic field on the vertical central axis [11].

On the horizontal central axis \( (r = \rho, \theta = \pi/2) \), equation (22) becomes

\[
B_{\rho}(\rho, \frac{\pi}{2}) = 0
\]

(27)

since the term \((P)_{r=\rho, \theta=\pi/2} + (Q)_{r=\rho, \theta=\pi/2} + (R)_{r=\rho, \theta=\pi/2}\) becomes zero.
Thus, on the horizontal central axis, the magnetic field has only the vertical components since only \( B_z \) values exist.

Similarly, equation (21) can be integrated using the integral table to obtain

\[
B_\theta(r, \theta) = \frac{\mu_0 n I a^2 \tan \theta}{4} \left[ \frac{P}{2} - Q - R \right]
\]  
(28)

On the vertical central axis \((r = z, \theta = 0)\), equation (28) becomes 0 because \( \tan \theta = 0 \)

\[
B_\theta(r = z, \theta = 0) = 0
\]  
(29)

and on the horizontal central axis \((r = \rho, \theta = \pi/2)\), equation (28) becomes

\[
B_\theta(r = \rho, \theta = \pi/2) = \frac{\mu_0 n I a^2 \tan(\pi/2)}{4} [0 - 0 - 0]
\]  
(30)

Equation (30) is an indeterminate form of \( \infty \times 0 \). So we substituted the limitation from \( \lim_{\theta \to \pi/2} \left[ \frac{P(\theta)}{2} - Q(\theta) - R(\theta) \right] \) to \( \lim_{\theta \to \pi/2} \frac{P(\theta) - Q(\theta) - R(\theta)}{\cot \theta} \) to differentiate the equation by applying l'Hôpital's rule, thereby obtaining

\[
B_\theta(r = \rho, \theta = \pi/2) = \frac{\mu_0 n I a^2}{4} \left[ -3\rho^4 + 5\rho^2 - \frac{2}{(\rho^2 + a^2)^{1/2}} \right] L
\]  
(31)

which is a function of \( \rho \).

Using equations (22) and (28), we can obtain the magnetic field \( \textbf{B}(r, \theta) \)

\[
\textbf{B}(r, \theta) = B_\rho(r, \theta) \hat{\rho} + B_\theta(r, \theta) \hat{\theta}
\]  
(32)

Thus, we can obtain the magnetic field of a solenoid at arbitrary point \((r, \theta)\). Equation (32) can be used to find the electromagnetic force acting on a magnet traversing a solenoid.

Meanwhile, to determine the \( \textbf{B}(\rho, z) \) for a solenoid, we can write equation (18) as

\[
dB_\rho = \frac{3\mu_0 n I a^2 \rho (z - z')}{4[\rho^2 + (z - z')^2 + a^2 + 2a\rho]^{3/2}} dz'
\]  
(33)

and integrate equation (33) after substituting \( a^2 + 2a\rho = c^2 \)

\[
B_\rho(\rho, z) = \frac{3\mu_0 n I a^2 \rho}{4} \left( \frac{1}{(s^2 + \rho^2 + c^2)^{3/2}} - \frac{1}{(t'^2 + \rho^2 + c^2)^{3/2}} \right)
\]  
(34)

where \( s = z - \frac{1}{2} \) and \( t' = z + \frac{1}{2} \). Because \( z \) is zero on the horizontal central axis, in equation (34), \( B_\rho = 0 \) (i.e., only \( B_z \) exists). From equation (19), we have

\[
dB_z = \frac{\mu_0 n I a^2 2[\rho^2 + (z - z')^2 + c^2] - 3\rho(\rho + a)}{[\rho^2 + (z - z')^2 + c^2]^{5/2}} dz'
\]  
(35)

This equation can be integrated using the integral table to obtain

\[
B_z(\rho, z) = \frac{\mu_0 n I a^2}{4} \left[ \frac{2[\rho^2 + (z - z')^2 + c^2] - 3\rho(\rho + a)}{[\rho^2 + (z - z')^2 + c^2]^{5/2}} dz' \right]
\]  
(36)

For the vertical central axis \((\rho = 0)\), we can derive from equation (36) as

\[
B_z(\rho = 0, z) = \frac{\mu_0 n I a^2}{4} \left[ \frac{2[2\rho^2 + (z - z')^2 + c^2] - 3\rho(\rho + a)}{[\rho^2 + (z - z')^2 + c^2]^{5/2}} dz' \right]
\]  
(37)

which is a well-known expression for the magnetic field on the vertical central axis of a solenoid [11].

Finally, we can obtain \( \textbf{B}(\rho, z) \) from equations (34) and (36):

\[
\textbf{B}(\rho, z) = B_\rho(\rho, z) \hat{\rho} + B_z(\rho, z) \hat{z}
\]  
(38)

Using this equation, we can determine the magnetic field at arbitrary point \((\rho, z)\). Equation (38) can also be used to obtain the electromagnetic force acting on a small magnet moving on the vertical central axis of a solenoid. As shown in equations (22), (28), (34), and (36), all components of magnetic fields of a finite solenoid depend on \( n, a, \) and \( L \). This dependence has been also reported in the earlier numerical studies [8,9].

### III. Results and discussion

While equations (22) and (28) are useful for determining the magnetic field at arbitrary point \((r, \theta)\), the equations in cylindrical coordinates are more helpful since
a solenoid has a cylindrical shape. Therefore, in this study, we used equations (34) and (36) for the estimation of the magnetic field around a solenoid. We performed a simulation using Wolfram Mathematica to determine the magnetic field on the horizontal and vertical axes of a solenoid, as shown in Fig. 3. The radius, length, number of turns, and applied current for the solenoid were $a = 2.5 \text{ cm}$, $L = 30 \text{ cm}$, $N = 600$, and $I = 1 \text{ A}$, respectively.

Figure 4 (a) shows $B_\rho (\rho, z)$ at the positions $z = L/4, L/2$, and $L$. Clearly, $B_\rho (\rho, z)$ becomes zero on the vertical central axes ($\rho = 0$). $B_\rho (\rho)$ shows relatively large values at the edge ($z = L/2$) of the solenoid and is very weak inside ($z = L/4$). But, the data for $z = L/4, L/2$ cannot represent exact magnetic fields $B_\rho (\rho)$ because it was not obtained within the approximate limit $r \gg a, a \gg r$, or $\sin \theta \approx 0$ (left side of the black dotted vertical line in Fig. 4(a)). In the circular coils and solenoids, $B_\rho (\rho)$ is expected to have a peak-like maximum of the magnetic field at the coil position (noted by an arrow in Fig. 4 (a)) but $B_\rho (\rho)$ is almost zero within the approximate limit. Figure 4 (b) shows $B_\rho (\rho)$ at the positions $\rho = a/50 (= 0.02a), a/20 (= 0.05a)$, and $4a/5 (= 0.8a)$. $B_\rho (\rho)$ at the positions $\rho = a/50$ and $a/20$ shows the peak-like narrow distributions at the edge positions (noted by two arrows in Fig. 4 (b)) while it shows the broad ones at the position $\rho = 4a/5$. These narrow field distributions at the edges have been reported in the earlier study [8] and represent the relatively exact fields whereas the broad ones cannot tell reliable magnetic values because the positions $\rho = 4a/5$ is not within the approximate limit, $r \gg a, a \gg r$, or $\sin \theta \approx 0$. Thus, equation (34) holds good for the regions near the $z$ axis or far away from a solenoid.

Figure 5 shows $B_z (z)$ at the positions $z = 0, a/50, a/20$, and $2a$. Evidently, on the vertical central axis, $B_z (\rho = 0, z)$ coincides with the known magnetic field distribution. For small $\rho$-values, $B_z (z)$ tends to very slightly decrease, compared to $B_z (\rho = 0, z)$. In Fig. 5, $B_z (z)$ inside the solenoid decreases by about 0.07 mT in each of three cases. It is noted that $B_z (\rho = a/50, z)$ approaches to $B_z (\rho = 0, z)$ on the vertical central axis at an error 4% and $B_z (\rho = a/20, z)$ is about 90% of $B_z (\rho = 0, z)$. In addition, as is well known, the magnetic fields at the edge (noted by an arrow in Fig. 5) of the solenoid become halves of those in the center in all cases.
This means that the derived formula equation (36) is a useful approximation for calculating near-axis magnetic fields of a finite solenoid even if they are not absolutely exact. Outside the solenoid, $B_z(z)$ is very small but not zero.

Figure 6 shows $B_z(\rho)$ at the positions $z = 0, L/4, L/2$ and $L$. $\rho$ values are selected within the approximate limit, $r \gg a, a \gg r$, or $\sin \theta \approx 0$, namely $\rho < a/20$. The slight variations of $B_z(\rho)$ according to $\rho$ probably originate from the approximate error or the natural properties of the finite solenoid as argued in previous articles [8,9]. It is noted that as is well known, $B_z(\rho, z = L/2)/B_z(\rho, z = 0)$ is 1/2 in Fig. 6.

Thus, we present useful analytic functions that can be used for determining the magnetic field at arbitrary points around a solenoid under the approximate conditions $r \gg a, a \gg r$, or $\sin \theta \approx 0$.

The magnetic fields should be measured for the conditions ($r \gg a, a \gg r$, or $\sin \theta \approx 0$) used for deriving the analytic function in this study to compare with the values obtained from the formula. Such measurements can be obtained by using a magnetic field sensor, which can be moved in the $\rho$- and $z$-direction. In the near future, we plan to conduct a study for experimentally measuring the magnetic field of a solenoid with an appropriate device.

IV. Conclusion

Using the magnetic field vector potential of a circular current loop and integrating it along the azimuthal direction, we derived approximate analytic functions for obtaining the magnetic field of a solenoid at an arbitrary off-axis points under the approximate conditions $r \gg a, a \gg r$, or $\sin \theta \approx 0$ of a finite solenoid. The derived analytic function of magnetic field reduces to a well-known magnetic field formula on the $z$-axis, showing the validity of the derived analytic function. All components of magnetic fields of a finite solenoid depend on $n, a$, and $L$.

The magnetic fields at arbitrary off-axis points within an approximate limit for a solenoid were estimated via simulations by using Wolfram Mathematica. A very weak and almost constant $B_\rho(\rho, z)$ was observed in the regions far away from the solenoid. Furthermore, $B_z(\rho, z)$ is not constant inside the solenoid, but very slightly decreases with respect to $\rho$ positions within the approximate limit and that the magnetic field is small but nonzero outside a finite-length solenoid. The reason for the small radical variation of $B_z(\rho, z)$ is unclear if the approximate error or the natural properties of the finite solenoid and experimental studies are needed for it.

In conclusion, we present the limited but useful analytic functions for determining the magnetic field at arbitrary points near the $z$ axis or far away from a solenoid. The analytic functions for the magnetic field derived in this study are expected to be useful for determining the magnetic field near the central $z$-axis of electromagnets and far the surface of permanent magnets, which are considered as solenoids, as well as the magnetic force acting on a magnet moving outside a solenoid.
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