A FREDHOLM MAPPING OF INDEX ZERO

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Abstract. Sufficient conditions are given to assert that between any two Banach spaces over $\mathbb{K}$ Fredholm mappings share exactly $N$ values in a specific open ball. The proof of the result is constructive and is based upon continuation methods.

1. Preliminaries

Let $X$ and $Y$ be two Banach spaces. If $F : X \to Y$ is a continuous mapping, then one way of solving the equation

$$F(x) = 0$$

is to embed (1) in a continuum of problems

$$H(x, t) = 0 \quad (0 \leq t \leq 1),$$

which is resolved when $t = 0$. When $t = 1$, the problem (2) becomes (1). In the case when it is possible to continue the solution for all $t$ in $[0, 1]$ then (1) is solved. This method is called continuation with respect to a parameter [1]-[23].

In this paper, sufficient conditions are given in order to prove that two differentiable mappings share exactly $N$ values in a specific open ball. Other conditions, sufficient to guarantee the existence of zero points in finite and infinite dimensional settings, have been given by the author in several other papers [10]-[23]. In this paper we use continuation methods. The proof supplies the existence of implicitly defined continuous mappings whose ranges reach zero points [5]-[7]. The key is the use of the Continuous Dependence theorem on a parameter in Banach spaces [25], properties of Fredholm $C^1$-mappings [25, 26], the Weierstrass theorem relative to extremum points [26], and a consequence of the properties of the algebra of Banach whose elements are the linear continuous mappings from a Banach space into itself.

We briefly recall some theorems and concepts to be used.

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Theorem 1 ([25, pp. 17-19] Continuous Dependence Theorem). Let the following conditions be satisfied:

(i) $P$ is a metric space, called the parameter space.
(ii) For each $p \in P$, the mapping $T_p$ satisfies the following hypotheses:

1. $T_p : M \subseteq X \rightarrow M$, i.e., $M$ is mapped into itself by $T_p$;
2. $M$ is a closed non-empty set in a complete metric space $(X, d)$;
3. $T_p$ is $k$-contractive for fixed $k \in [0, 1]$.

(iii) For a fixed $p_0 \in P$, and for all $x \in M$, $\lim_{p \rightarrow p_0} T_p(x) = T_{p_0}(x)$.

Thus, for each $p \in P$, the equation $x_p = T_p x_p$ has exactly one solution, where $x_p \in M$ and $\lim_{p \rightarrow p_0} x_p = x_{p_0}$.

Definition ([25, 26]). We will assume $X$ and $Y$ are Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Mapping $F : D(F) \subseteq X \rightarrow Y$, is said to be compact whenever it is continuous and the image $F(B)$ is relatively compact (i.e., its closure $\overline{F(B)}$ is compact in $Y$) for every bounded subset $B \subset D(F)$.

Mapping $F$ is said to be proper whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

If $D(F)$ is open, then mapping $F$ is said to be a Fredholm mapping if and only if both $F$ is a $C^1$-mapping and $F'(x) : X \rightarrow Y$ is a Fredholm linear mapping for all $x \in D(F)$. That $L : X \rightarrow Y$ is a linear Fredholm mapping means that $L$ is linear and continuous and both the numbers $\dim(\ker(L))$ and $\codim(R(L))$ are finite, and therefore $\ker(L) = X_1$ is a Banach space and has topological complement $X_2$, since $\dim(X_1)$ is finite. The integer number $\dim(L) = \dim(\ker(L)) - \dim(R(L))$ is called the index of $L$, where $\dim$ signifies dimension, $\dim$ codimension, $\ker$ kernel and $R(L)$ stands for the range of mapping $L$.

Let $\mathcal{F}(X, Y)$ denote the set of all linear Fredholm mappings $A : X \rightarrow Y$. Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L : X \rightarrow Y$. Let $\text{Isom}(X, Y)$ denote the set of all isomorphisms $L : X \rightarrow Y$.

Let $B(x_0, \rho)$ denote the open ball of centre $x_0$ and radius $\rho$, and $S(x_0, \rho)$ the sphere of centre $x_0$ and radius $\rho$. If $u : X \rightarrow Y$ is a linear continuous bijective operator, the inverse linear continuous operator to $u$ will be denoted by $u^{-1}$.

Theorem 2 ([27, pp. 23-24]). (a) The set $\text{Isom}\mathcal{L}(X, Y)$ is open in $\mathcal{L}(X, Y)$.

(b) The mapping $\beta : \text{Isom}(X, Y) \rightarrow \mathcal{L}(X, Y)$, $\beta(u) := u^{-1}$ is continuous.

Theorem 3 ([26, p. 296]). Let $g : D(g) \subset X \rightarrow Y$ be a compact mapping, where $a \in D(g)$. If the derivative $g'(a)$ exists, then $g'(a) \in \mathcal{L}(X, Y)$ is also a compact mapping.

Theorem 4 ([26, p. 366]). Let $S \in \mathcal{F}(X, Y)$. The perturbed mapping $S + C$ verifies $S + C \in \mathcal{F}(X, Y)$ and $\dim(S + C) = \dim(S)$ if $C \in \mathcal{L}(X, Y)$ and $C$ is a compact mapping.

Definition ([26, p. 318]). Let $F : X \rightarrow Y$ be a $C^1$-mapping.
The point $u \in X$ is called a **regular point** of $F$ if and only if $F'(u) \in \mathcal{L}(X,Y)$ maps onto $Y$, and $\ker(F'(x))$ splits $X$ into a topological direct sum.

The point $v \in Y$ is called a **regular value** of $A$ if and only if the pre-image $F^{-1}(v)$ is empty or consists solely of regular points.

2. A Fredholm mapping

If we can say $u := f - g$, then $u$ has a zero if and only if $f$ and $g$ share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We thereby establish our result in terms of $f, g$.

**Theorem 5.** Let $f, g : X \to Y$ be two $C^1$-mappings, where $X$ and $Y$ are two Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(i) $f$ is a proper and Fredholm mapping of index zero and $g$ is a compact mapping.

(ii) Mapping $f$ has $N$ zeros, $x_i, i = 1, \ldots, N$ in $B(x_0, \rho)$.

(iii) Zero is a regular value of the mapping $f(\cdot) - tg(\cdot) : X \to Y$ for each parameter $t \in [0, 1]$.

(iv) If $(x, t) \in S(x_0, \rho) \times [0, 1]$ then $f(x) \neq tg(x)$.

Hence the following statement holds true:

(a) $f$ and $g$ share exactly $N$ values in the open ball $B(x_0, \rho)$.

**Proof.** (a) Henceforth $X \times \mathbb{R}$ is provided by the topology, given by the product norm. $\mathcal{L}(X,Y), \mathcal{L}(Y,X)$ are provided by the topologies given by their respective operator norm.

(a1) Let us construct the following homotopy $H : X \times [0, 1] \to Y$, $H(x, t) := f(x) - tg(x)$, which is a $C^1$-homotopy between the mappings $f$ and $f - g$.

Henceforth partial derivatives will generally be denoted by writing initial spaces as subindices of mappings.

We will see here for any $(x, t) \in X \times [0, 1]$ that $H_x(x, t) = f'(x) - tg'(x)$ verifies $H_x(x, t) \in \mathcal{F}(X,Y)$, and $\text{ind}H_x(x, t) = 0$.

Since $g$ is a compact mapping and the derivative $g'(x)$ exists for any fixed $x \in X$, Theorem 3 implies that $g'(x) \in \mathcal{L}(X,Y)$ is a compact mapping and therefore for any $(x, t) \in X \times [0, 1]$, $tg'(x) \in \mathcal{L}(X,Y)$, is also a compact mapping.

Since $f$ is a Fredholm mapping of index zero, then $f'(x) \in \mathcal{F}(X,Y), \forall x \in X$ and $\text{ind}(f'(x)) = 0, \forall x \in X$.

These results together with Theorem 4 imply that $H_x(x, t) \in \mathcal{F}(X,Y)$, and $\text{ind}H_x(x, t) = 0, \forall (x, t) \in X \times [0, 1]$.

(a2) We will now prove that, if $H(x, t) = 0, (x, t) \in B(x_0, \rho) \times [0, 1]$, then $H_x(x, t) \in \text{isom}(X,Y)$.

Let $(x, t) \in X \times [0, 1], H(x, t) = 0$. Since zero is a regular value of $f(x) - tg(x)$, therefore $H_x(x, t)$ maps onto $Y$, therefore $\text{codim}(R(H_x(x, t))) = \dim(Y/Y) = 0$ and hence $\text{ind}(H_x(x, t)) = \dim(\ker(H_x(x, t)))$. 

Furthermore, since \( \text{ind}(H_x(x,t)) = 0 \), therefore \( \dim(\ker(H_x(x,t))) = 0 \), and hence \( H_x(x,t) \) is also injective. Thus, \( H_x(x,t) \) is a bijective linear continuous mapping, and since \( Y \) is a Banach space, the linear inverse mapping \( H_x(x,t)^{-1} \in \mathcal{L}(Y,X) \) is also continuous. Hence, \( H_x(x,t) \in \text{Isom}(X,Y) \).

(a3) We will prove the existence of a compact set \( V'' \) which contains all \( x \) in \( X \) such that \( f(x) - tg(x) = 0 \), when \( (x,t) \in B(x_0,\rho) \times [0,1] \). Let us define the set \( V := g(D) \), where

\[
D := \{ x \in B(x_0,\rho) : \exists t \in [0,1], \ t = t(x), \ \text{such that} \ f(x) = tg(x) \}.
\]

Since \( f(x_i) = 0 = f(x_i) - 0g(x_i), \ i = 1,\ldots,N, \) therefore \( x_i \in D, \ i = 1,\ldots,N, \) hence \( D \) is not empty. Owing to \( V \subset g(B(x_0,\rho)) \), we know that \( V \) is a bounded set, and with \( g \) as a compact mapping, then \( V \) is a relatively compact set.

We now construct the set \( V' := \{ ty : t \in [0,1], \ y \in V \} \). \( V' \) is a compact set in \( Y \) due to the fact that it can be written in the following way \( V' = v(\overline{V} \times [0,1]) \), where \( v \) is the continuous mapping \( v : \overline{V} \times [0,1] \to Y, \ v(y,t) = ty \), and \( \overline{V} \times [0,1] \) is a compact set in the topological product space \( Y \times \mathbb{R} \).

Since \( f \) is a proper mapping and \( V' \) is a compact set on \( Y \), the pre-image of \( V' \) under \( f \), \( V'' := f^{-1}(V') \) is a compact set on \( X \), which contains all \( x \) which verify the following \( f(x) - tg(x) = 0, \ (x,t) \in B(x_0,\rho) \times [0,1] \).

(a4) We will prove that there is a real number \( C > 0 \) such that if

\[
(x,t) \in (H^{-1}\{0\}) \cap (B(x_0,\rho) \times [0,1]),
\]

where \( H^{-1}\{0\} \) is the pre-image of zero under \( H \), then \( \|H_x(x,t)^{-1}\| \leq C \), where \( H_x(x,t)^{-1} \) is the inverse mapping of \( H_x(x,t) \).

Since \( H \) is a \( C^1 \)-mapping, the mapping \( H : X \times \mathbb{R} \to \mathcal{L}(X,Y), \ (x,t) \mapsto H_x(x,t) \) is continuous. From (a2) if \( (x,t) \) belongs to \( (H^{-1}\{0\}) \cap (B(x_0,\rho) \times [0,1]) \), then \( H_x(x,t) \in \text{Isom}(X,Y) \). From Theorem 2, the mapping inverse formation \( \beta \) : \( \text{Isom}(X,Y) \subset \mathcal{L}(X,Y) \to \mathcal{L}(Y,X), \beta(u) = u^{-1} \), is a continuous mapping. Consequently, by composition of continuous mappings, the mapping

\[
\| \circ \beta \circ H_x : H^{-1}\{0\} \cap (V'' \times [0,1]) \subset X \times \mathbb{R} \to \mathbb{R}, \ (x,t) \mapsto \|H_x(x,t)^{-1}\|,
\]

is continuous.

Since \( H : X \times [0,1] \) is a continuous mapping, \( H^{-1}\{0\} \subset X \times [0,1] \) is a closed set, and as \( V'' \times [0,1] \subset X \times \mathbb{R} \) is a compact set, therefore \( H^{-1}\{0\} \cap (V'' \times [0,1]) \subset X \times \mathbb{R} \) is a compact set. Weierstrass Theorem implies that there is maximum of \( \|H_x(x,t)^{-1}\| \) when \( (x,t) \in H^{-1}\{0\} \cap (V'' \times [0,1]) \), and hence, there is a real number \( C > 0 \), such that \( \|H_x(x,t)^{-1}\| \leq C, \forall (x,t) \in (H^{-1}\{0\}) \cap (V'' \times [0,1]) \).

(a5) Let suppose that \( (x_a,t_a) \in B(x_0,\rho) \times [0,1] \) and that \( H(x_a,t_a) = 0 \). Therefore:

From (a3), \( (x_a,t_a) \in V'' \times [0,1] \).

From (a2), \( H_x(x_a,t_a) \in \text{Isom}(X,Y) \).
We will now prove the existence of \( r_0 > 0, r > 0 \) and the existence a continuous mapping \( x(\cdot) : [t_a - r_0, t_a + r_0] \cap [0, 1] \to X \), which verifies
\[
\| x(t) \| < r, H(x_a + x(t), t) = 0, \forall t \in [t_a - r_0, t_a + r_0] \cap [0, 1].
\]

To this end, we define \( G(x, t) := H(x_a + x(t), t), \forall x \in X \), and we solve the equation
\[
G(x, t) = 0
\]
for \( x \). Obviously, we have \( G(0, t_a) = H(x_a, t_a) = 0 \), and furthermore, \( G_x(0, t_a) \) verifies \( G_x(0, t_a) = H_x(x_a, t_a) \).

We transform Equation (3) into the following equivalent equation:
\[
H_x(x_a, t_a)^{-1} [H_x(x_a, t_a)(x) - G(x, t)] = x.
\]
Equation (4) leads us to define the two following mappings
\[
h(x, t) := H_x(x_a, t_a)(x) - G(x, t),
\]
and
\[
T_t(x) := H_x(x_a, t_a)^{-1} ((h(x, t)),
\]
where \( h \) is a \( C^1 \)-mapping, and
\[
h(0, t_a) = 0.
\]
Equation (4) is equivalent to the following “key equation”
\[
T_t(x) = x.
\]
Let us observe that \( t \) in the definition of \( T_t \) is an index and not a partial derivative as is usually written. Equation (3) is equivalent to the Fixed Point Equation (6), which will be studied below.

Let \( x, x' \in B(x_0, \rho); t, t_a \in [0, 1] \) such that \( |t - t_a|, \| x \|, \| x' \| < r, |t - t_a| < r_0 \), where \( r, r_0 \) will be fixed at a later stage.

Since \( h_x(x, t) = H_x(x_a, t_a) - G_x(x, t) \), hence
\[
h_x(0, t_a) = 0.
\]
From Equation (7) and since \( h_x : X \times [0, 1] \to \mathcal{L}(X, Y), (x, t) \mapsto h_x(x, t) \) is continuous, the Taylor theorem implies that
\[
\| h(x,t) - h(x', t) \| \leq \sup \{ \| h_x((x' + \theta(x - x'), t) : \theta \in [0, 1]) \| x - x' \| \}
= o(1) \| x - x' \|, \quad o(1) \to 0 \quad \text{as} \quad r \to 0.
\]
Due to Equations (5) and (8), and since \( h \) is a continuous mapping, therefore
\[
\| h(x, t) \| \leq \| h(x, t) - h(0, t) \| + \| h(0, t) \| = o(1) \| x \| + o'(1),
\]
o(1) \to 0 \quad \text{as} \quad r \to 0, \quad o'(1) \to 0 \quad \text{as} \quad r_0' \to 0.

Hence
\[
\| T_t(x) \| \leq \| H_x(x_a, t_a)^{-1} \| \| h(x, t) \| \leq \| H_x(x_a, t_a)^{-1} \| (o(1) \| x \| + o'(1)),
\]
o(1) \to 0 \quad \text{as} \quad r \to 0, \quad o'(1) \to 0 \quad \text{as} \quad r_0' \to 0.
Now $r$ is fixed so that $o(1) \leq \frac{1}{2C}$, and then the closed and non-empty set $M := \{x \in X : \|x\| \leq r\}$ is constructed. We are now able to fix $r_0$, so that $o'(1) < \frac{1}{2C}$, and the set $M' := \{t \in [0,1] : \|t - t_o\| \leq \min\{r_r, r_0\} = r_0\}$ is constructed. We prove below that the hypotheses of Theorem 1 are verified by the spaces and mappings, we have just defined.

The Metric Space $(M', |\cdot|)$ will be considered as the parameter space of the hypothesis (i) of Continuous Dependence Theorem 1. $M$ will be considered as the closed and non-empty set and $(X, \|\cdot\|)$ as the complete metric space considered in hypothesis (ii) of Theorem 1, which is verified as we will see in the two following paragraphs.

Owing to Equation (9), for any fixed $t \in M'$, and for all $x \in M$,

$$\|T_t(x)\| \leq \|H_x(x, t_o)\| o(1) \|x\| + o'(1) \leq C(\frac{1}{2C}r + \frac{1}{2C}r) \leq r,$$

therefore $T_t(x) \in M$, and hence $T_t : M \to M$. That is, $T_t$ maps the closed non-empty set $M$ of the Banach space $X$ into itself.

Due to Equation (8), for any $x, x' \in M$ and all fixed $t \in M'$

$$\|T_t(x) - T_t(x')\| \leq \|H_x(x, t_o)\|^{-1} \|h(x, t) - h(x', t)\| \leq \|H_x(x, t_o)\|^{-1} \|x - x'\| \leq \frac{1}{2} \|x - x'\|,$$

therefore $T_t$ is half-contractive for any $t \in M'$ which has been fixed. Hence hypothesis (ii) of Theorem 1 is verified.

For any fixed $t_0 \in M'$ and for all $x \in M$,

$$\lim_{t \to t_0} T_t(x) = \lim_{t \to t_0} H_x(x, t_o)^{-1}(H_x(x, t_o)(x) - G(x, t)) = H_x(x, t_o)^{-1}(H_x(x, t_o)(x) - G(x, t_0)) = T_{t_0}(x),$$

and hence hypothesis (iii) of Theorem 1 is also verified.

Thus, Theorem 1 implies, for any $t \in M'$, that $T_1$ has a unique fixed point $T_1(x) = x := x(t)$, and it is verified that $x(t) \to x(t_0)$ as $t \to t_0$, $t, t_0 \in M'$, that is, $x(\cdot)$ is a continuous mapping. Thus for each $t \in M'$ there is only one $x(t) \in M \subset X$ such that $G(x(t), t) = 0$, and hence

$$H(x_a + x(t), t) = 0.$$

Let us also observe that $T_{t_0}(0) = 0$, $x(t_0) = 0$. Equation (10) can be written in the following way: $H(\alpha(t), t) = 0, \forall t \in M'$, where $\alpha$ is the following continuous mapping $\alpha : M' \to Y$, $\alpha(t) := x_a + x(t)$.

\textbf{(a6)} We will now prove that $f$ and $g$ share exactly $N$ values in the open ball $B(x_0, \rho)$, using iteratively the process of the previous section a finite number of times, to find each shared value. To this end we have to prove that it is possible to select the same $r_0$ for each iteration of the process of the previous section.

Let us define the mapping

$$\varphi : V'' \times [0,1] \times V'' \times [0,1] \subset X \times [0,1] \times X \times [0,1] \to Y,$$
\[ \varphi(x_a, t_a; x, t) := H_x(x_a, t_a)x - H(x_a + x, t), \]

which, as a composition of continuous mappings, is uniformly continuous on the compact set \[ V'' \times [0, 1] \times V'' \times [0, 1] \] of the product topological space \( X \times R \times X \times R \). Therefore for any fixed \( r > 0 \), there is \( \delta(\frac{r}{2C}) > 0 \) such that, if \( (x_a, t_a; x, t), (x_{a'}, t_{a'}; x', t') \in V'' \times [0, 1] \times V'' \times [0, 1] \), with \( \| (x_a, t_a; x, t) - (x_{a'}, t_{a'}; x', t') \| < \delta(\frac{r}{2C}) \), then \( \| \varphi(x_a, t_a; x, t) - \varphi(x_{a'}, t_{a'}; x', t') \| < \frac{r}{2C} \).

If we restrict the domain of mapping \( \varphi \) by fixing any \((x_a, t_a) \in V'' \times [0, 1] \) such that \( H(x_a, t_a) = 0 \), we obtain mapping \( h \) considered in the previous section, that is

\[ h : (H^{-1}\{0\}) \cap (V'' \times [0, 1]) \subset X \times R \rightarrow Y, \]

\[ b(x, t) = \varphi(x_a, t_a; x, t) = H_x(x_a, t_a)(x) - H(x_a + x, t). \]

We are now able to fix \( r'_0 \) considered in the previous section by taking \( r'_0 = \delta(\frac{r}{2C}) \), where \( r \) will be established later in this section.

On the other hand the mapping \( \varphi_x : \ V'' \times [0, 1] \times V'' \times [0, 1] \rightarrow L(X, Y), \)

\[ \varphi_x(x_a, t_a; x, t) = H_x(x_a, t_a) - H_x(x_a + x, t), \]

is uniformly continuous on the compact set \( V'' \times [0, 1] \times V'' \times [0, 1] \), and therefore there is \( \delta(\frac{r}{2C}) > 0 \) such that, if \( \varphi(x_a, t_a; x, t), (x_{a'}, t_{a'}; x', t') \in V'' \times [0, 1] \times V'' \times [0, 1] \), then

\[ \| (x_a, t_a; x, t) - (x_{a'}, t_{a'}; x', t') \| < \delta(\frac{1}{2C}) \]

\[ \Rightarrow \| \varphi_x(x_a, t_a; x, t) - \varphi_x(x_{a'}, t_{a'}; x', t') \| < \frac{1}{2C}. \]

Let us observe that the mapping \( h_x \) considered in the previous section is the mapping \( \varphi_x \), when \( (x_a, t_a) \) is fixed: \( h_x : V'' \times [0, 1] \rightarrow L(X, Y), \)

\[ h_x(x, t) = \varphi_x(x_a, t_a; x, t) = H_x(x_a, t_a) - H_x(x_a + x, t). \]

At this point we determine the previously mentioned \( r \) by taking \( r = \delta(\frac{1}{2C}) \).

Since \( r \) and \( r'_0 \) have been fixed, we are now able to fix \( r_0 \) in the same way as in the previous section, that is, \( r_0 = \min\{r, r'_0\} \).

The previous section implies that if \( H(x_a, t_a) = 0, (x_a, t_a) \in B(x_0, \rho) \times [0, 1] \) then there is a continuous mapping \( x(\cdot) : [t_a - r_0, t_a + r_0] \cap [0, 1] \rightarrow X \), which verifies

\[ H(x_a + x(t), t) = 0, \forall t \in [t_a - r_0, t_a + r_0] \cap [0, 1]. \]

This lets us construct the continuous mapping \( \alpha : [t_a, t_a + r_0] \rightarrow Y, \alpha(t) = x_a + x(t) \) with \( H(\alpha(t), t) = 0, \forall t \in [t_a, t_a + r_0], \alpha(t_a) = x_a \).

We repeat this process of \( (a5) \) by taking \( (\alpha(t_a + r_0), t_a + r_0) \) as an initial point in each iteration, where \( (x_a, t_a) \) is the previous initial point, and \( (x_a, t_a) \in B(x_0, \rho) \times [0, 1], i = 1, \ldots, N \) as the initial point of the first iteration with \( \alpha : [0, r_0] \rightarrow Y, \alpha(t) = x_i + x(t), \alpha(0) = x_i \), to be extended in successive iterations of the process. A point \( (x_i', 1) \in B(x_0, \rho) \times [0, 1] \) which verifies \( H(x_i', 1) = 0, i \in 1, \ldots, N \) is reached in a finite number of iterations, since \( [0, 1] \) is a compact set, and from the frontier condition established in hypothesis (iv) of the theorem.

In an identical way, but starting at in a value shared by \( f \) and \( g \), and by the same process but taking initial successive points conveniently, we reach a zero
of \( f \) on the ball \( B(x_0, \rho) \). Therefore \( f \) has the same number of zeros that shared values by \( f \) and \( g \).

\[ \square \]

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