ON THE CENTROID OF THE PRIME GAMMA RINGS

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Abstract. We define and study the extended centroid of a prime Γ-ring.

1. Introduction


2. Preliminaries

Let \( M \) and \( \Gamma \) be two abelian groups. If for all \( x, y, z \in M \) and all \( \alpha, \beta \in \Gamma \) the conditions

(i) \( x\alpha y \in M \),

(ii) \( (x + y)\alpha z = x\alpha z + y\alpha z \), \( x(\alpha + \beta)z = x\alpha z + x\beta z \), \( x\alpha(y + z) = x\alpha y + x\alpha z \),

(iii) \( (x\alpha y)\beta z = x\alpha(y\beta z) \)

are satisfied, then we call \( M \) a Γ-ring. By a right (resp. left) ideal of a Γ-ring \( M \) we mean an additive subgroup \( U \) of \( M \) such that \( U \Gamma M \subseteq U \) (resp. \( M\Gamma U \subseteq U \)). If \( U \) is both a right and a left ideal, then we say that \( U \) is an ideal of \( M \). For each \( a \) of a Γ-ring \( M \) the smallest right ideal containing \( a \) is called the principal right ideal generated by \( a \) and

Received September 8, 1999. Revised May 11, 2000.

2000 Mathematics Subject Classification: 16N60, 16W25, 16Y99.

Key words and phrases: extended centroid, symmetric bi-derivation, trace, quotient Γ-ring.
is denoted by $<a>_r$. Similarly we define $<a>_l$ (resp. $<a>$), the principal left (resp. two sided) ideal generated by $a$. An ideal $P$ of a $\Gamma$-ring $M$ is said to be prime if for any ideals $A$ and $B$ of $M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal $Q$ of a $\Gamma$-ring $M$ is said to be semi-prime if for any ideal $U$ of $M$, $UTU \subseteq Q$ implies $U \subseteq Q$. A $\Gamma$-ring $M$ is said to be prime (resp. semi-prime) if the zero ideal is prime (resp. semi-prime).

**Theorem 2.1.** ([2, Theorem 4]) If $M$ is a $\Gamma$-ring, the following conditions are equivalent:

(i) $M$ is a prime $\Gamma$-ring.

(ii) If $a, b \in M$ and $a\Gamma M\Gamma b = (0)$, then $a = 0$ or $b = 0$.

(iii) If $<a>$ and $<b>$ are principal ideals in $M$ such that $<a>_\Gamma <b>= (0)$, then $a = 0$ or $b = 0$.

(iv) If $A$ and $B$ are right ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.

(v) If $A$ and $B$ are left ideals in $M$ such that $A\Gamma B = (0)$, then $A = (0)$ or $B = (0)$.

3. Centroids

Let $M$ be a $\Gamma$-ring. A mapping $D(\cdot, \cdot): M \times M \to M$ is said to be symmetric bi-additive if it is additive in both arguments and $D(x, y) = D(y, x)$ for all $x, y \in M$. By the trace of $D(\cdot, \cdot)$ we mean a map $d: M \to M$ defined by $d(x) = D(x, x)$ for all $x \in M$. A symmetric bi-additive map is called a symmetric bi-derivation if $D(x_\alpha z, y) = D(x, y)\beta z + x_\beta D(z, y)$ for all $x, y, z \in M$ and $\beta \in \Gamma$. Since a map $D(\cdot, \cdot)$ is symmetric bi-additive, the trace of $D(\cdot, \cdot)$ satisfies the relation $d(x + y) = d(x) + d(y) + 2D(x, y)$ for all $x, y \in M$ and is an even function.

Let $M$ be a prime $\Gamma$-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U, f) \mid U(\neq 0) \text{ is an ideal of } M \text{ and } f: U \to M \text{ is a right } M\text{-module homomorphism}\}.$$

Define a relation $\sim$ on $\mathcal{M}$ by $(U, f) \sim (V, g) \iff \exists W(\neq 0) \subset U \cap V$ such that $f = g$ on $W$. Since $M$ is a prime $\Gamma$-ring, it is possible to find a
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non-zero $W$ and so "~" is an equivalence relation. This gives a chance for us to get a partition of $M$. We then denote the equivalence class by $Cl(U, f) = \hat{f}$, where $\hat{f} := \{g : V \to M|(U, f) \sim (V, g)\}$, and denote by $Q$ the set of all equivalence classes. Now we define an addition "+" on $Q$ as follows:

$$\hat{f} + \hat{g} = Cl(U, f) + Cl(V, g) = Cl(U \cap V, f + g)$$

where $f + g : U \cap V \to M$ is a right $M$-module homomorphism. Assume that $(U_1, f_1) \sim (U_2, f_2)$ and $(V_1, g_1) \sim (V_2, g_2)$. Then $\exists W_1(\neq 0) \subset U_1 \cap U_2$ such that $f_1 = f_2$; and $\exists W_2(\neq 0) \subset V_1 \cap V_2$ such that $g_1 = g_2$. Taking $W = W_1 \cap W_2$. Then $W \neq 0$ and

$$W = W_1 \cap W_2 \subset (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2).$$

For any $x \in W$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ in $W$. Therefore $(U_1 \cap V_1, f_1 + g_1) \sim (U_2 \cap V_2, f_2 + g_2)$, which means that the addition "+" is well-defined.

Now we will prove that $Q$ is an additive abelian group. Let $\hat{f} = Cl(U, f)$, $\hat{g} = Cl(V, g)$ and $\hat{h} = Cl(W, h)$ be elements of $Q$. Then

$$(\hat{f} + \hat{g}) + \hat{h} = Cl(U \cap V, f + g) + Cl(W, h)$$
$$= Cl((U \cap V) \cap W, (f + g) + h)$$
$$= Cl(U \cap (V \cap W), f + (g + h))$$
$$= Cl(U, f) + Cl(V \cap W, g + h)$$
$$= \hat{f} + (\hat{g} + \hat{h}).$$

Taking $\hat{0} := Cl(M, 0)$ where $0 : M \to M$, $x \mapsto 0$, for all $x \in M$ we have $\hat{f} + \hat{0} = Cl(U, f) + Cl(M, 0) = Cl(U \cap M, f + 0) = Cl(U, f) = \hat{f}$, and similarly $\hat{0} + \hat{f} = \hat{f}$. Hence $\hat{0}$ is the additive identity in $Q$. For any element $\hat{f} = Cl(U, f)$ of $Q$, it is easy to show that $-\hat{f} = Cl(U, -f)$ is an additive inverse of $f = Cl(U, f)$. Finally, for any elements $\hat{f} = Cl(U, f)$
and \( \hat{g} = Cl(V, g) \) of \( Q \), we have

\[
\hat{f} + \hat{g} = Cl(U, f) + Cl(V, g) \\
= Cl(U \cap V, f + g) \\
= Cl(V \cap U, g + f) \\
= Cl(V, g) + Cl(U, f) \\
= \hat{g} + \hat{f}.
\]

Therefore \( Q \) is an additive abelian group.

Since \( MTM \neq M \) and since \( M \) is a prime \( \Gamma \)-ring, \( MTM \neq 0 \) is an ideal of \( M \). We can take the homomorphism \( 1_{TM} : MTM \to M \) as a unit \( M \)-module homomorphism. Note that \( M\beta M \neq 0 \) for all \( 0 \neq \beta \in \Gamma \) so that \( 1_{M\beta} : M\beta M \to M \) is non-zero \( M \)-module homomorphism. Denote

\[
\mathcal{N} := \{(M\beta M, 1_{M\beta}) | 0 \neq \beta \in \Gamma\},
\]

and define a relation \( \sim \) on \( \mathcal{N} \) by \( (M\beta M, 1_{M\beta}) \sim (M\gamma M, 1_{M\gamma}) \iff \exists W := M\alpha M(\neq 0) \subset M\beta M \cap M\gamma M \) such that \( 1_{M\beta} = 1_{M\gamma} \) on \( W \). We can easily check that \( \sim \) is an equivalence relation on \( \mathcal{N} \). Denote by \( Cl(M\beta M, 1_{M\beta}) = \hat{\beta} \), the equivalence class containing \( (M\beta M, 1_{M\beta}) \) and by \( \hat{\Gamma} \) the set of all equivalence classes of \( \mathcal{N} \) with respect to \( \sim \), that is,

\[
\hat{\beta} := \{1_{M\gamma} : M\gamma M \to M | (M\beta M, 1_{M\beta}) \sim (M\gamma M, 1_{M\gamma})\}
\]

and \( \hat{\Gamma} := \{\hat{\beta} | 0 \neq \beta \in \Gamma\} \). Define an addition \( + \) on \( \hat{\Gamma} \) as follows:

\[
\hat{\beta} + \hat{\delta} = Cl(M\beta M, 1_{M\beta}) + Cl(M\delta M, 1_{M\delta}) \\
= Cl(M\beta M \cap M\delta M, 1_{M\beta} + 1_{M\delta})
\]

for every \( \beta(\neq 0), \delta(\neq 0) \in \Gamma \). Then \( (\hat{\Gamma}, +) \) is an abelian group. Now we define a mapping \( (-, -, -) : Q \times \hat{\Gamma} \times Q \to Q, (\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g} \), as follows:

\[
\hat{f}\hat{\beta}\hat{g} = Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g) \\
= Cl(V\Gamma M\beta M U, f1_{M\beta}g)
\]
where
\[ V\Gamma M\beta M\Gamma U = \left\{ \sum v_i \gamma_i m_i \beta n_i \alpha_i u_i \mid v_i \in V, u_i \in U, m_i, n_i \in M \text{ and } \alpha_i, \gamma_i \in \Gamma \right\} \]
is an ideal of \( M \) and \( f1_{M\beta g} : V\Gamma M\beta M\Gamma U \to M \) which is given by
\[ f1_{M\beta g}\left( \sum v_i \gamma_i m_i \beta n_i \alpha_i u_i \right) = f\left( \sum g(v_i) \gamma_i m_i \beta n_i \alpha_i u_i \right) \]
is a right \( M \)-module homomorphism. Then it is routine to check that such mapping is well-defined. We will show that \( Q \) is a \( \hat{\Gamma} \)-ring with unity. Let \( \hat{f}, \hat{g}, \hat{h} \in Q \) and \( \hat{\beta}, \hat{\gamma} \in \hat{\Gamma} \), i.e., \( \hat{f} = Cl(U, f), \hat{g} = Cl(V, g), \hat{h} = Cl(W, h), \hat{\beta} = Cl(M\beta M, 1_{M\beta}) \) and \( \hat{\gamma} = Cl(M\gamma M, 1_{M\gamma}) \). Then
\[
(\hat{f} + \hat{g})\hat{\beta}\hat{h} = (Cl(U, f) + Cl(V, g))Cl(M\beta M, 1_{M\beta})Cl(W, h)
= Cl(U \cap V, f + g)Cl(M\beta M, 1_{M\beta})Cl(W, h)
= Cl(W\Gamma M\beta M\Gamma(U \cap V), (f + g)1_{M\beta}h)
= Cl(W\Gamma M\beta M\Gamma \cap W\Gamma M\beta M\Gamma V, f1_{M\beta}h + g1_{M\beta}h)
= Cl(W\Gamma M\beta M\Gamma U, f1_{M\beta}h) + Cl(W\Gamma M\beta M\Gamma V, g1_{M\beta}h)
= \hat{f}\hat{\beta}\hat{h} + \hat{g}\hat{\beta}\hat{h},
\]
and the equalities \( \hat{f}(\hat{\gamma} + \hat{\beta})\hat{g} = \hat{f}\hat{\gamma}\hat{g} + \hat{f}\hat{\beta}\hat{g} \) and \( \hat{f}\hat{\beta}(\hat{g} + \hat{h}) = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\beta}\hat{h} \) are proved in an analogous way. Moreover we have
\[
(\hat{f}\hat{\gamma}\hat{g})\hat{\beta}\hat{h} = (Cl(U, f)Cl(M\gamma M, 1_{M\gamma})Cl(V, g))Cl(M\beta M, 1_{M\beta})Cl(W, h)
= Cl(V\Gamma M\gamma M\Gamma U, f1_{M\gamma}g)Cl(M\beta M, 1_{M\beta})Cl(W, h)
= Cl(W\Gamma M\beta M\Gamma (V\Gamma M\gamma M\Gamma U), (f1_{M\gamma}g)1_{M\beta}h)
= Cl((W\Gamma M\beta M\Gamma V\Gamma M\gamma M\Gamma U), f1_{M\gamma}(g1_{M\beta}h))
= Cl(U, f)Cl(M\gamma M, 1_{M\gamma})Cl(W\Gamma M\beta M\Gamma V, g1_{M\beta}h)
= Cl(U, f)Cl(M\gamma M, 1_{M\gamma})(Cl(V, g)Cl(M\beta M, 1_{M\beta})Cl(W, h))
= \hat{f}\hat{\gamma}(\hat{g}\hat{\beta}\hat{h}).\]
Next we will show that $Q$ has an identity. Let $\hat{f} \in Q$ and $\hat{\beta} \in \hat{\Gamma}$. Take $\hat{I} = \text{Cl}(M, I) \in Q$ where $I : M \to M$, $x \mapsto x$, is a $M$-module homomorphism. Then

$$
\hat{f} \hat{\beta} \hat{I} = \text{Cl}(U, f)\text{Cl}(M\beta M, 1_{M\beta})\text{Cl}(M, I)
$$

$$
= \text{Cl}(MTM\beta M\Gamma U, f1_{M\beta I})
$$

$$
= \text{Cl}(U, f) = \hat{f},
$$

and similarly we have $\hat{I} \hat{\beta} \hat{f} = \hat{f}$. Hence $Q$ is a $\hat{\Gamma}$-ring with identity. Noticing that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$ is an isomorphism, we know that the $\hat{\Gamma}$-ring $Q$ is a $\Gamma$-ring. Finally we prove that $M$ is a subring of $Q$. For a fixed element $a$ in $M$ and every element $\gamma \in \Gamma$, consider a mapping $\lambda_{\alpha\gamma} : M \to M$ defined by $\lambda_{\alpha\gamma}(x) = a\gamma x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{\alpha\gamma}$ is a right $M$-module homomorphism, so that $\lambda_{\alpha\gamma}$ is an element of $Q$. Define a mapping $\psi : M \to Q$ by $\psi(a) = \hat{a} = \text{Cl}(M, \lambda_{\alpha\gamma})$ for all $a \in M$ and $\gamma \in \Gamma$. Clearly $\psi$ is well-defined. To prove $\psi$ is one-to-one, it is enough to show that

$$
\ker\psi = \{a \in M \mid \psi(a) = \hat{0}\} = \{0_M\}.
$$

Let $a \in \ker\psi$. Then $\psi(a) = \hat{0}$, i.e., $\text{Cl}(M, \lambda_{\alpha\gamma}) = \text{Cl}(M, 0)$. It follows that $0_M = \lambda_{\alpha\gamma}(M) = a\gamma M$. Since $M$ is a prime $\Gamma$-ring, we have $a = 0_M$ and so $\ker\psi = \{0_M\}$. In order to prove $\psi$ is a homomorphism, let $\gamma, \beta \in \Gamma$ and $a, b \in M$. Then

$$
\lambda_{(a+b)\gamma}(x) = (a + b)\gamma x = a\gamma x + b\gamma x
$$

$$
= \lambda_{a\gamma}(x) + \lambda_{b\gamma}(x) = (\lambda_{a\gamma} + \lambda_{b\gamma})(x)
$$

and

$$
\lambda_{(ab)\gamma}(x) = (ab)\gamma x = a\beta(b\gamma x) = \lambda_{a\beta}(b\gamma x)
$$

$$
= \lambda_{a\beta}(1_{M\beta}(b\gamma x)) = \lambda_{a\beta}(1_{M\beta}(\lambda_{b\gamma}(x)))
$$

$$
= (\lambda_{a\beta}1_{M\beta}(b\gamma))(x)
$$

for all $x \in M$. It follows that $\lambda_{(a+b)\gamma} = \lambda_{a\gamma} + \lambda_{b\gamma}$ and $\lambda_{(ab)\gamma} =
\[ \lambda_{a\beta}1_{M\beta}\lambda_{b\gamma}. \] Hence
\[
\psi(a + b) = \overline{a + b} = Cl(M, \lambda_{(a+b)\gamma})
= Cl(M \cap M, \lambda_{a\gamma} + \lambda_{b\gamma})
= Cl(M, \lambda_{a\gamma}) + Cl(M, \lambda_{b\gamma})
= \hat{a} + \hat{b} = \psi(a) + \psi(b)
\]

and
\[
\psi(a\beta b) = \overline{a\beta b} = Cl(M, \lambda_{(a\beta b)\gamma})
= Cl(M\Gamma M\beta M\Gamma M, \lambda_{a\beta}1_{M\beta}\lambda_{b\gamma})
= Cl(M, \lambda_{a\beta})Cl(M\beta M, 1_{M\beta})Cl(M, \lambda_{b\gamma})
= \hat{a}\beta\hat{b}
= \psi(a)\beta\psi(b). \quad [\Gamma \text{ is isomorphic to } \hat{\Gamma}].
\]

Therefore \( M \) is a subring of \( Q \), and in such case we call \( Q \) the quotient \( \Gamma \)-ring of \( M \).

Let \( M \) be any \( \Gamma \)-ring (in the sense of Barnes) and let \( E(M, \Gamma) \) be the set of endomorphisms of the additive group of \( M \). We can easily check that \( E(M, \Gamma) \) is a \( \Gamma \)-ring. For \( a \in M \), define maps \( R_a : M \to M \) and \( L_a : M \to M \) by \( R_a(m) = m\gamma a \) and \( L_a(m) = a\gamma m \), respectively, for all \( m \in M \) and \( \gamma \in \Gamma \). Then \( R_a, L_a \in E(M, \Gamma) \). Let \( B(M, \Gamma) \) be the subring of \( E(M, \Gamma) \) generated by all \( R_a \) and \( L_a \) for \( a \in M \).

**Definition 3.1.** The set of elements in \( E(M, \Gamma) \) which commute elementwise with \( B(M, \Gamma) \) is called the centroid of \( M \).

For purposes of convenience, we use \( q \) instead of \( \hat{q} \in Q \).

**Lemma 3.2.** Let \( M \) be a prime \( \Gamma \)-ring. For each non-zero \( q \in Q \), there is a non-zero ideal \( U \) of \( M \) such that \( q(U) \subseteq M \).

**Proof.** Straightforward.

**Lemma 3.3.** Let \( M \) be a prime \( \Gamma \)-ring. Then the quotient \( \Gamma \)-ring \( Q \) of \( M \) is a prime \( \Gamma \)-ring.
Proof. Let \( p, q \in Q \) be such that \( p\Gamma Q\Gamma q = 0 \). If \( p \neq 0 \neq q \), then there exist non-zero ideals \( U \) and \( V \) of \( M \) such that \( p(U) \subset M \) and \( q(V) \subset M \). Since \( p \neq 0 \neq q \), there exist non-zero elements \( u \in U \) and \( v \in V \) such that \( p(u) \neq 0 \neq q(v) \). Noticing that \( M \) is a subring of \( Q \), we have

\[
p(u)\Gamma M\Gamma q(v) \subset p(u)\Gamma Q\Gamma q(v) = 0
\]

and so \( p(u)\Gamma M\Gamma q(v) = 0 \). This is a contradiction. Hence \( p = 0 \) or \( q = 0 \), ending the proof.

**Definition 3.4.** The set

\[
C_\Gamma := \{ g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma \}
\]

is called the **extended centroid** of a \( \Gamma \)-ring \( M \).

Let \( M \) be a prime \( \Gamma \)-ring and let \( C_\Gamma \) be the extended centroid of \( M \). Note that if \( a_i \) and \( b_i \) are non-zero elements of \( M \) such that \( \sum a_i\gamma_i x\beta_i b_i = 0 \) for all \( x \in M \) and \( \beta_i, \gamma_i \in \Gamma \), then the \( a_i \)'s (also \( b_i \)'s) are linearly dependent over \( C_\Gamma \). Moreover, if \( a\gamma x\beta b = b\gamma x\beta a \) for all \( x \in M \) and \( \beta, \gamma \in \Gamma \) where \( a(\neq 0), b \in M \) are fixed, then there exists \( \lambda \in C_\Gamma \) such that \( b = \lambda a a \) for \( a \in \Gamma \).

**Lemma 3.5.** Let \( M \) be a 2-torsion free prime \( \Gamma \)-ring, \( D(\cdot, \cdot) \) the symmetric bi-derivation of \( M \) and \( d \) the trace of \( D(\cdot, \cdot) \). If

\[
a\gamma d(x) = 0
\]

for all \( x \in M \) and \( \gamma \in \Gamma \) where \( a \) is a fixed element of \( M \), then \( a = 0 \) or \( D = 0 \).

Proof. Let \( x, y, z \in M \) and \( \beta, \gamma \in \Gamma \). Replacing \( x \) by \( x + y \) in (1), we get

\[
a\gamma D(x, y) = 0.
\]

If we substitute \( z\beta x \) for \( x \) in (2), then

\[
a\gamma z\beta D(x, y) = 0.
\]

Since \( M \) is a prime \( \Gamma \)-ring, it follows that \( a = 0 \) or \( D = 0 \).
LEMMA 3.6. Let \( M \) be a 2-torsion free prime \( \Gamma \)-ring, \( D_1(\cdot, \cdot) \) and \( D_2(\cdot, \cdot) \) the symmetric bi-derivations of \( M \) and \( d_1 \) and \( d_2 \) the traces of \( D_1(\cdot, \cdot) \) and \( D_2(\cdot, \cdot) \), respectively. If
\[
d_1(x)\gamma d_2(y) = d_2(x)\gamma d_1(y)
\]
for all \( x, y \in M \) and \( \gamma \in \Gamma \) and \( d_1 \neq 0 \), then there exists \( \lambda \in C_\Gamma \) such that \( d_2(x) = \lambda \alpha d_1(x) \) for \( \alpha \in \Gamma \), where \( C_\Gamma \) is the extended centroid of \( M \).

Proof. Let \( x, y, z \in M \) and \( \beta, \gamma \in \Gamma \). Substituting \( y + z \) for \( y \) in (4), we have
\[
d_1(x)\gamma D_2(y, z) = d_2(x)\gamma D_1(y, z).
\]
Replacing \( z \) by \( z\beta y \) in (5), we have
\[
d_1(x)\gamma z\beta d_2(y) = d_2(x)\gamma z\beta d_1(y).
\]
Now if we replace \( y \) by \( x \) in (6), then
\[
d_1(x)\gamma z\beta d_2(x) = d_2(x)\gamma z\beta d_1(x).
\]
If \( d_1(x) \neq 0 \) then \( d_2(x) = \lambda(x)\alpha d_1(x) \) for all \( \alpha \in \Gamma \) and for some \( \lambda(x) \in C_\Gamma \). Thus if \( d_1(x) \neq 0 \neq d_1(y) \), then it follows from (6) that
\[
(\lambda(y) - \lambda(x))\alpha d_1(x)\gamma z\beta d_1(y) = 0.
\]
Since \( M \) is a prime \( \Gamma \)-ring, by using Lemma 3.5 we conclude that \( \lambda(x) = \lambda(y) \). Hence we have proved that there exists \( \lambda \in C_\Gamma \) such that \( d_2(x) = \lambda \alpha d_1(x) \) for all \( \alpha \in \Gamma \) and \( x \in M \) with \( d_1(x) \neq 0 \). On the other hand, if \( d_1(x) = 0 \) then \( d_2(x) = 0 \) as well. Therefore \( d_2(x) = \lambda \alpha d_1(x) \) for all \( x \in M \) and \( \alpha \in \Gamma \).

THEOREM 3.7. Let \( M \) be a 2-torsion free prime \( \Gamma \)-ring, \( D_1(\cdot, \cdot), D_2(\cdot, \cdot), D_3(\cdot, \cdot) \) and \( D_4(\cdot, \cdot) \) the symmetric bi-derivations of \( M \) and \( d_1, d_2, d_3 \) and \( d_4 \) the traces of \( D_1(\cdot, \cdot), D_2(\cdot, \cdot), D_3(\cdot, \cdot) \) and \( D_4(\cdot, \cdot) \) respectively. If
\[
d_1(x)\gamma d_2(y) = d_3(x)\gamma d_4(y)
\]
for all \( x, y \in M \) and \( \gamma \in \Gamma \) and \( d_1 \neq 0 \neq d_4 \), then there exists \( \lambda \in C_\Gamma \) such that \( d_2(x) = \lambda \alpha d_4(x) \) and \( d_3(x) = \lambda \alpha d_1(x) \) for \( \alpha \in \Gamma \) where \( C_\Gamma \) is the extended centroid of \( M \).
Proof. Let \( x, y, z, w \in M \) and \( \alpha, \beta, \gamma \in \Gamma \). Replacing \( y \) by \( y + z \) in (9), we get

\[
(10) \quad d_1(x)\gamma D_2(y, z) = d_3(x)\gamma D_4(y, z).
\]

If we substitute \( z\beta x \) for \( z \) in (10), then

\[
(11) \quad d_1(x)\gamma z\beta d_2(y) = d_3(x)\gamma z\beta d_4(y).
\]

Substituting \( z\alpha d_4(w) \) for \( z \) in (11), we have

\[
(12) \quad d_1(x)\gamma z\alpha d_4(w)\beta d_2(y) = d_3(x)\gamma z\alpha d_4(w)\beta d_4(y).
\]

By (11), we know that \( d_1(x)\gamma z\alpha d_2(w) = d_3(x)\gamma z\alpha d_4(w) \) and so

\[
d_1(x)\gamma z\alpha (d_4(w)\beta d_2(y) - d_2(w)\beta d_4(y)) = 0
\]

which implies that \( d_4(w)\beta d_2(y) = d_2(w)\beta d_4(y) \) since \( d_1 \neq 0 \) and \( M \) is a prime \( \Gamma \)-ring. It follows from \( d_4 \neq 0 \) and Lemma 3.6 that \( d_2(y) = \lambda \alpha d_4(y) \) for some \( \lambda \in C_\Gamma \). Hence, by (11), we conclude that

\[
(\lambda \alpha d_1(x) - d_3(x))\gamma z\beta d_4(y) = 0,
\]

and so \( d_3(x) = \lambda \alpha d_1(x) \). This completes the proof. \( \square \)

References

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