CONTRACTIONS OF CLASS $Q$ AND INVARIANT SUBSPACES

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Abstract. A Hilbert Space operator $T$ is of class $Q$ if $T^2T^2 - 2T^*T + I$ is nonnegative. Every paranormal operator is of class $Q$, but class-$Q$ operators are not necessarily normaloid. It is shown that if a class-$Q$ contraction $T$ has no nontrivial invariant subspace, then it is a proper contraction. Moreover, the nonnegative operator $Q = T^2T^2 - 2T^*T + I$ also is a proper contraction.

1. Introduction

Let $\mathcal{H}$ be a nonzero complex Hilbert space. By a subspace $M$ of $\mathcal{H}$ we mean a closed linear manifold of $\mathcal{H}$, and by an operator $T$ on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. A subspace $M$ is invariant for $T$ if $T(M) \subseteq M$, and nontrivial if $\{0\} \neq M \neq \mathcal{H}$. Let $\mathcal{B}[\mathcal{H}]$ denote the algebra of all operators on $\mathcal{H}$. For an arbitrary operator $T$ in $\mathcal{B}[\mathcal{H}]$ set, as usual, $|T| = (T^*T)^{1/2}$ (the absolute value of $T$) and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$ (the self-commutator of $T$), where $T^*$ is the adjoint of $T$, and consider the following standard definitions: $T$ is hyponormal if $[T^*, T]$ is nonnegative (i.e., $|T^*|^2 \leq |T|^2$; equivalently, $\|T^*x\| \leq \|T^2x\|$ for every $x$ in $\mathcal{H}$), $T$ is of class $U$ if $|T|^2 - |T^2|$ is nonnegative (i.e., $|T|^2 \leq |T^2|$), paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x$ in $\mathcal{H}$, and normaloid if $r(T) = \|T\|$ (where $r(T)$ denotes the spectral radius of $T$). These are related by proper inclusion:

- Hyponormal $\subset$ Class $U$ $\subset$ Paranormal $\subset$ Normaloid.

A contraction is an operator $T$ such that $\|T\| \leq 1$ (i.e., $\|Tx\| \leq \|x\|$ for every $x$ in $\mathcal{H}$; equivalently, $T^*T \leq I$). A proper contraction is an operator $T$ such that $\|Tx\| < \|x\|$ for every nonzero $x$ in $\mathcal{H}$ (equivalently,
A strict contraction is an operator $T$ such that $\|T\| < 1$ (i.e., $\sup_{0 \neq x}(\|Tx\|/\|x\|) < 1$ or, equivalently, $T^*T < I$, which means that $T^*T \leq \gamma I$ for some $\gamma \in (0, 1)$). Again, these are related by proper inclusion: Strict Contraction $\subset$ Proper Contraction $\subset$ Contraction.

It was recently proved in [10] that if a hyponormal contraction $T$ has no nontrivial invariant subspace, then $T$ is a proper contraction and its self-commutator $[T^*, T]$ is a strict contraction. This was extended in [5] to contractions of class $U$ (if a contraction $T$ in $U$ has no nontrivial invariant subspace, then both $T$ and the nonnegative operator $\|T^2\| - |T|^2$ are proper contractions), and to paranormal contractions in [6]: If a paranormal contraction $T$ has no nontrivial invariant subspace, then $T$ is a proper contraction and so is the nonnegative operator $\|T^2\|^2 - 2|T|^2 + I$. In the present paper we extend this result to contractions of class $Q$. Operators of class $Q$ are defined below. This is a class of operators that properly includes the paranormal operators.

## 2. Operators of class $Q$

In this section we define operators of class $Q$ and consider some basic properties, examples and counterexamples, in order to put this class in its due place. Recall that, for any real $\lambda$ and any operator $T \in B[\mathcal{H}]$

$$\lambda \|T^2x\| \|x\| \leq \frac{1}{2}(\|T^2x\|^2 + \lambda^2 \|x\|^2)$$

and, in particular, for $\lambda = 1$,

$$\|T^2x\| \|x\| \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2),$$

for every $x \in \mathcal{H}$. An operator $T \in B[\mathcal{H}]$ is paranormal if

$$\|Tx\|^2 \leq \|T^2x\| \|x\|$$

for every $x \in \mathcal{H}$. Paranormal operators have been much investigated since [8] (see e.g., [7] and [9]). The following alternative definition is well-known. An operator $T \in B[\mathcal{H}]$ is paranormal if and only if

$$O \leq T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I$$

for all $\lambda > 0$ (cf. [1], also see [12]). Equivalently, $T$ is paranormal if and only if

$$\lambda \|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \lambda^2 \|x\|^2)$$

for every $x \in \mathcal{H}$, for all $\lambda > 0$. Note that the above inequalities hold trivially for every $\lambda \leq 0$ for all operators $T \in B[\mathcal{H}]$. Take any operator $T$ in $B[\mathcal{H}]$ and set
\[
Q = T^{2*}T^2 - 2T^*T + I.
\]

**Definition 1.** An operator \( T \) is of class \( Q \) if \( O \leq Q \). Equivalently, \( T \in Q \) if
\[
\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)
\]
for every \( x \).

Since \( O \leq T^{2*}T^2 - 2T^*T + \lambda^2I \) if and only if \( \lambda^{-\frac{1}{2}}T \in Q \) for any \( \lambda > 0 \),
\( T \) is paranormal if and only if \( \lambda T \in Q \) for all \( \lambda > 0 \).

Every paranormal operator is a normaloid of class \( Q \). That is, with \( N \) and \( P \) standing for the classes of all normaloid and paranormal operators from \( \mathcal{B}[\mathcal{H}] \), respectively, it is clear that
\[
P \subseteq Q \cap N.
\]
However, \( Q \not\subseteq N \) and \( Q \cap N \not\subseteq P \). Indeed, \( S = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q \) for every \( \lambda \in (0, 1/\sqrt{2}] \) but \( S \not\in N \) (nonzero nilpotent) for all \( \lambda \neq 0 \). Moreover, \( T = I \oplus S \) lies in \((Q \cap N) \setminus P \) for any \( \lambda \in (0, 1/\sqrt{2}] \). In fact, \( S \) is not normaloid, and hence not paranormal, which implies that \( T \) is not paranormal (restriction of a paranormal to an invariant subspace is again paranormal), and \( r(T) = \|T\| = 1 \). Thus \( T \) is a normaloid contraction of class \( Q \) that is not paranormal.

**Proposition 1.** Let \( T \in \mathcal{B}[\mathcal{H}] \) be an operator of class \( Q \).

(a) The restriction of \( T \) to an invariant subspace is again a class-\( Q \) operator.

(b) If \( T \) is invertible, then \( T^{-1} \) is of class \( Q \).

**Proof.** Let \( T \) be an operator of class \( Q \) and let \( M \) be a \( T \)-invariant subspace.

(a) If \( u \in M \), then
\[
2\|T|_M u\|^2 = 2\|Tu\|^2 \leq \|T^2u\|^2 + \|u\|^2
\]
and so \( T|_M \) is of class \( Q \).

(b) If \( T \) is invertible, then
\[
2\|x\|^2 = 2\|TT^{-1}x\|^2 \leq \|T^2(T^{-1}x)\|^2 + \|T^{-1}x\|^2
\]
for every \( x \in \mathcal{H} \). Take any \( y \) in \( \mathcal{H} = \text{ran}(T) \) so that \( y = Tx \), \( x = T^{-1}y \) and \( T^{-1}x = T^{-2}y \) for some \( x \) in \( \mathcal{H} \). Thus
\[
2\|T^{-1}y\|^2 \leq \|y\|^2 + \|T^{-2}y\|^2
\]
by the above inequality, and so \( T^{-1} \) is of class.

Some properties that the paranormal operators inherit from the hyponormals survive up to class \( Q \), as in the case of Proposition 1. However, many important properties shared by the hyponormals do not travel well up to class \( Q \). For instance, there exist nonzero quasinilpotent operators of class \( Q \) (a quasinilpotent normaloid is obviously null),
compact operators of class $\mathcal{Q}$ that are not normal (every compact paranormal is normal [11]), and also operators of class $\mathcal{Q}$ for which isolated points of the spectrum are not eigenvalues (isolated points of the spectrum of a paranormal are eigenvalues [2]). Here is an example. The compact weighted unilateral shift $T = \text{shift}(\{ \frac{1}{k+1} \}_{k=1}^{\infty})$ is a quasinilpotent ($r(T) = 0$) contraction ($\|T\| = \frac{1}{2}$) with no eigenvalues ($0$ is in the residual spectrum of $T$). Clearly, since $T$ is not normaloid, it is not paranormal. But it is of class $\mathcal{Q}$. Indeed,

$$O < \text{diag}(\{ 1 - \frac{2}{(k+1)^2} \}_{k=1}^{\infty}) = I - 2T^*T < T^{2*}T^2 - 2T^*T + I.$$  

Another common property of hyponormal and paranormal operators that does not apply to class $\mathcal{Q}$ is that a multiple of a class-$\mathcal{Q}$ operator may not be of class $\mathcal{Q}$. For example, $S = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{Q}$ for every $\lambda \in (0, 1/\sqrt{2}]$, but $S \notin \mathcal{Q}$ for all $\lambda > 1/\sqrt{2}$. Actually, $\mathcal{Q}$ is not a cone in $\mathcal{B}[\mathcal{H}]$, although its intersection with the closed unit ball is balanced (a subset $A$ of a linear space is balanced if $\alpha A \subseteq A$ whenever $|\alpha| \leq 1$).

**Proposition 2.** Let $T$ be a Hilbert space operator.

(a) If $\|T\| \leq 1/\sqrt{2}$, then $T \in \mathcal{Q}$.
(b) If $T^2 = O$, then $T \in \mathcal{Q}$ if and only if $\|T\| \leq 1/\sqrt{2}$.
(c) If $T \in \mathcal{Q}$, $T^2 \neq O$ and $|\alpha| \leq \min\{1, \|T^2\|^{-1}\}$, then $\alpha T \in \mathcal{Q}$.

In particular, if $T \in \mathcal{Q}$ is a contraction, then $\alpha T \in \mathcal{Q}$ whenever $|\alpha| \leq 1$.

(d) A contraction $T$ in $\mathcal{Q}$ is paranormal if and only if $O \leq T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I$ for all $\lambda \in (0, 1)$.

**Proof.** Let $T$ be any operator in $\mathcal{B}[\mathcal{H}]$.

(a) Since $O \leq I - 2T^*T$ (that is, $2T^*T \leq I$) if and only if $\alpha T$ for $|\alpha| = \sqrt{2}$ is a contraction, it follows that $\|\sqrt{2}T\| \leq 1$ implies $T \in \mathcal{Q}$ because

$$I - 2T^*T \leq T^{2*}T^2 - 2T^*T + I.$$

(b) If $T^2 = O$, then $T \in \mathcal{Q}$ if and only if $O \leq I - 2T^*T$.

(c) If $T$ lies in $\mathcal{Q}$, then

$$2|\alpha|^2T^*T \leq |\alpha|^2T^{2*}T^2 + |\alpha|^2I$$

and hence, for every scalar $\alpha$,

$$2|\alpha|^2T^*T - |\alpha|^4T^{2*}T^2 - I \leq (1 - |\alpha|^2)(|\alpha|^2T^{2*}T^2 - I).$$
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Suppose $T^2 \neq O$. Note: $|\alpha| \leq \|T^2\|^{-1}$ (i.e., $\alpha T^2$ is a contraction) if and only if $|\alpha|^2 T^2 + T^2 \leq I$. If, in addition, $|\alpha| \leq 1$, then $(1 - |\alpha|^2) (|\alpha|^2 T^2 - I) \leq O$, and therefore $\alpha T \in \mathcal{Q}$.

(d) If $T \in \mathcal{Q}$ is a contraction, then $\alpha T$ lies in $\mathcal{Q}$ for all $\alpha \in (0, 1]$ or, equivalently (with $\lambda = \alpha^{-1}$), $O \leq T^2 T - 2 \lambda T^* T + \lambda^2 I$ for all $\lambda \geq 1$. Thus, if $T \in \mathcal{Q}$ is a contraction, then the above inequality holds for all $\lambda > 0$ if and only if it holds for all $\lambda \in (0, 1)$. Therefore, a contraction $T$ of class $\mathcal{Q}$ is paranormal if and only if $O \leq T^2 T - 2 \lambda T^* T + \lambda^2 I$ for all $\lambda \in (0, 1)$.

**Corollary 1.** If $T \in \mathcal{Q}$ is invertible, then $\alpha T \in \mathcal{Q}$ for every scalar $\alpha$ such that either $|\alpha| \leq \min\{1, \|T^2\|^{-1}\}$ or $|\alpha| \geq \max\{1, \|T^{-2}\|\}$.

**Proof.** Take an invertible $T \in \mathcal{Q}$ and any scalar $\alpha$. Proposition 2 ensures that $\alpha T \in \mathcal{Q}$ whenever $|\alpha| \leq \min\{1, \|T^2\|^{-1}\}$, and Proposition 1 says that $T^{-1} \in \mathcal{Q}$. Then $\beta T^{-1} \in \mathcal{Q}$ for every nonzero scalar $\beta$ such that $|\beta| \leq \min\{1, \|T^{-2}\|^{-1}\}$ by Proposition 2. Put $\gamma = \beta^{-1}$ so that $(\gamma T)^{-1}$ lies in $\mathcal{Q}$ for each scalar $\gamma$ such that $|\gamma|^{-1} \leq \min\{1, \|T^{-2}\|^{-1}\}$; equivalently, such that $|\gamma| \geq \max\{1, \|T^{-2}\|\}$. Therefore, applying Proposition 1 again, it follows that $\gamma T \in \mathcal{Q}$ whenever $|\gamma| \geq \max\{1, \|T^{-2}\|\}$, which completes the proof.

If $T$ is an invertible operator in $\mathcal{Q}$ and $\min\{1, \|T^2\|^{-1}\} = \max\{1, \|T^{-2}\|\}$, then the above corollary ensures that $T$ is paranormal. In particular, if $T$ is an invertible contraction in $\mathcal{Q}$ for which the above min and max coincide, then $T$ is an invertible paranormal contraction; a unitary operator, actually, as we shall see in Proposition 3 below (every invertible contraction for which the above min and max coincide is unitary). Note that there exist invertible normaloid contractions in $\mathcal{Q}$ that are not unitary so that the above min and max do not coincide. For instance, a weighted bilateral shift with increasing positive weights in $(1/2, 1)$ is a nonunitary invertible hyponormal contraction, thus paranormal, and so a normaloid of class $\mathcal{Q}$.

**Proposition 3.** If $T$ is an invertible contraction and
\[
\min\{1, \|T^n\|^{-1}\} = \max\{1, \|T^{-n}\|\}
\]
for some positive integer $n$, then $T$ is unitary.
Proof. Take any positive integer \( n \). If \( T \) is an invertible operator, then so is \( T^n \). If \( \|T\| \leq 1 \), then \( \|T^n\|^{-1} \geq 1 \) and hence \( \min\{1, \|T^n\|^{-1}\} = 1 \). But \( 1 \leq \|T^{-n}\| \|T^n\| \), and so \( \|T^{-n}\| \geq 1 \), which implies that \( \max\{1, \|T^{-n}\|\} = \|T^{-n}\| \). If \( n \) is unitary (reason: \( \|T^n\| \leq 1 \), and an invertible operator \( U \) such that \( U \) and \( U^{-1} \) are both contractions must be unitary). But if \( T \) is a contraction and \( T^n \) is an isometry, then \( T \) is an isometry. Indeed, if \( T \) is a contraction, then so is \( T(n-1) \), which means that \( T^{*(n-1)}T^{(n-1)} \leq I \), and therefore

\[
I = T^{*n}T^n = T^*(T^{*(n-1)}T^{(n-1)})T \leq T^*T \leq I
\]

so that \( T \) is an isometry. Dually, if \( T \) is a contraction and \( T^n \) is a coisometry, then \( T \) is a coisometry. Thus, if \( T \) contraction and \( T^n \) unitary, then \( T \) unitary.

Proposition 4. Suppose \( T \) is an operator of class \( Q \).

(a) If \( T^2 \) is a contraction, then so is \( T \).

(b) If \( T^2 \) is an isometry, then \( T \) is paranormal.

Proof. Let \( T \in B[H] \) be an operator of class \( Q \).

(a) Observe that \( T \) is of class \( Q \) if and only if

\[
2(T^*T - I) \leq T^*T - I.
\]

Thus \( T^*T \leq I \) implies \( T^*T \leq I \); that is, \( T \) is a contraction whenever \( T^2 \) is.

(b) Take any \( x \) in \( H \) and note that \( T \) is of class \( Q \) if and only if

\[
2\|Tx\|^2 \leq (\|T^2x\| - \|x\|)^2 + 2\|T^2x\|\|x\|.
\]

Hence \( \|T^2x\| = \|x\| \) implies \( \|Tx\|^2 \leq \|T^2x\|\|x\| \), for every \( x \in H \).

Therefore, if \( T \) is an operator of class \( Q \) for which \( T^2 \) is an isometry, then \( T \) is a paranormal contraction. Since \( T^*T = I \) implies \( Q = 2(I-T^*T) \), it follows that if \( T^2 \) is an isometry, then \( T \in Q \) if and only if \( T \) is a contraction and, in this case, \( T \) is paranormal.

Note that the converses fail. For instance, the weighted unilateral shift \( T = \text{shift}(2, \frac{1}{2}, 2, \frac{1}{2}, \cdots) \) is such that \( T^2 \) coincides with the square of the “unweighted” unilateral shift. Thus \( T^2 \) is an isometry, but \( T \) is not a contraction (\( \|T\| = 2 \)), and hence \( T \notin Q \) by Proposition 4 (so that \( T \) is not paranormal — in fact, \( T \) is not even normaloid: \( r(T) = 1 \)).

A part of an operator is a restriction of it to an invariant subspace. An operator \( T \) is hereditarily normaloid if every part of it is normaloid,
and totally hereditarily normaloid if it is hereditarily normaloid and every invertible part of it has a normaloid inverse [3]. The class of all hereditarily normaloid operators from $B[\mathcal{H}]$ is denoted by $\mathcal{HN}$, and the class of all totally hereditarily normaloid operators from $\mathcal{HN}$ is denoted by $T\mathcal{HN}$. Recall that (see e.g., [4]) \[ P \subset T\mathcal{HN} \subset \mathcal{HN} \subset \mathcal{N}. \]

Let $\mathcal{M}$ be any invariant subspace for $T$. Proposition 1 ensures that the following assertions hold true.

(a) If $T \in \mathcal{Q} \cap \mathcal{HN}$, then $T|_{\mathcal{M}} \in \mathcal{Q} \cap \mathcal{HN}$.
(b) If $T \in \mathcal{Q} \cap T\mathcal{HN}$ then $T|_{\mathcal{M}} \in \mathcal{Q} \cap T\mathcal{HN}$ and, if $T|_{\mathcal{M}}$ is invertible, then $(T|_{\mathcal{M}})^{-1} \in \mathcal{Q} \cap \mathcal{N}$.

Note that $T = I \oplus S$, with $S = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for any $\lambda \in (0, 1/\sqrt{2}]$, is a contraction in $(\mathcal{Q} \cap \mathcal{N}) \setminus \mathcal{HN}$. In fact, $S$ is not normaloid so that $T$ is not in $\mathcal{HN}$. There are two ways for an operator $T$ to be in $T\mathcal{HN}$: either $T \in \mathcal{HN}$ has no invertible part, or it has invertible parts and all of them have a normaloid inverse. The latter case prompts the question: are the invertible operators in $\mathcal{Q} \cap T\mathcal{HN}$ paranormal? More generally, is it true that, if $T$ is an invertible normaloid operator with a normaloid inverse, then $T \in \mathcal{Q}$ implies $T \in P$? (i.e., $T \in \mathcal{Q}$ implies $\lambda T \in \mathcal{Q}$ for all $\lambda > 0$?)

3. Invariant subspace theorem for contractions of class $\mathcal{Q}$

Take any operator $T$ in $B[\mathcal{H}]$ and set $D = I - T^*T$. Recall that $T$ is a contraction if and only if $D$ is nonnegative. In this case, $D^{1/2}$ is the defect operator of $T$.

**Proposition 5.** A contraction $T$ lies in $\mathcal{Q}$ if and only if $\|D^{1/2}Tx\| \leq \|D^{1/2}x\|$ for every $x$ in $\mathcal{H}$.

**Proof.** For any $T \in B[\mathcal{H}]$ put $Q = T^2 - 2T^*T + I$ and $D = I - T^*T$. Since

$$ Q = D - T^*DT, $$

it follows that $O \leq Q$ if and only if $\langle T^*DTx; x \rangle \leq \langle Dx; x \rangle$ for every $x \in \mathcal{H}$ or, equivalently, $\|D^{1/2}Tx\|^2 \leq \|D^{1/2}x\|^2$ for every $x \in \mathcal{H}$ if $T$ is a contraction. \qed
If a contraction $T$ has no nontrivial invariant subspace, then $D$ is a proper contraction. Indeed, if $T$ is a contraction with no nontrivial invariant subspace, then $\ker(T) = \{0\}$ so that $\|D^{1/2}x\|^2 = \|x\|^2 - \|Tx\|^2 < \|x\|^2$ for every nonzero $x$ in $\mathcal{H}$, which means that $D^{1/2}$ (and so $D$) is a proper contraction. If, in addition, $T$ is of class $Q$, then more is true.

**Theorem 1.** If a contraction $T \in Q$ has no nontrivial invariant subspace, then both $T$ and $Q$ are proper contractions.

**Proof.** Let $T \neq O$ be a contraction of class $Q$. Since $\ker(D) = \ker(D^{1/2})$, it follows by Proposition 5 that $\ker(D)$ is an invariant subspace for $T$. Suppose $T$ has no nontrivial invariant subspace so that either $\ker(D) = \mathcal{H}$ or $\ker(D) = \{0\}$. In the former case $D = O$; that is, $T^*T = I$, and so $T$ is an isometry, which is a contradiction: isometries have nontrivial invariant subspaces. In the latter case $D > O$; that is, $T^*T < I$, which means that $T$ is a proper contraction. Moreover, if $T$ is a contraction of class $Q$, then the nonnegative operator $Q$ is such that the power sequence $\{Q^n\}_{n \geq 1}$ converges strongly to $P$ (i.e., $Q^n \xrightarrow{s} P$), where $P$ is an orthogonal projection, and $TP = O$ so that $PT^* = O$ ($P$ is self-adjoint) [6]. If $T$ has no nontrivial invariant subspace, then $T^*$ has no nontrivial invariant subspace as well. Since $\ker(P)$ is a nonzero invariant subspace for $T^*$ whenever $PT^* = O$ and $T \neq O$, it follows that $\ker(P) = \mathcal{H}$. Hence $P = O$, and therefore $Q^n \xrightarrow{s} O$; that is, the nonnegative operator $Q$ is strongly stable. But strong stability coincides with proper contractiveness for quasinormal operators [6]; in particular, for nonnegative operators. Thus $Q$ also is a proper contraction. \qed

**References**

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