Some Characterizations of Modules via Essentially Small Submodules

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Abstract. In this paper, the structure of \( e \)-local modules and classes of modules via essentially small are investigated. We show that the following conditions are equivalent for a module \( M \):

1. Introduction

Throughout this paper, \( R \) will be an associative ring with identity and all modules are unitary \( R \)-module. We write \( M_{R} \) (resp., \( R_{M} \)) to indicate that \( M \) is a right (resp., left) \( R \)-module. All modules are right unital unless stated otherwise. If \( N \) is a submodule of \( M \), we denote by \( N \leq M \). Moreover, we write \( N \leq_{e} M \) and \( N \leq_{\oplus} M \) to indicate that \( N \) is an essential submodule, a direct summand and a small submodule of \( M \), respectively. If \( X \) is a subset of a right \( R \)-module \( M \), the
right annihilator of $X$ in $R$ is denoted by $r_R(X)$ or simply $r(X)$ if no confusion appears.

Recently, some authors have studied generalizations of semiperfect rings and perfect rings via projectivity of modules and small submodules of modules see [7, 11, 16, 18, 19]. Following [19], a submodule $N$ of $M$ is called $\delta$-small in $M$ (denote $N \ll_\delta M$) if $M = N + L$ and $M/L$ singular then $L = M$. In [7], the author extends the definition of lifting and supplemented modules to what he calls $\delta$-lifting and $\delta$-supplemented. This extension is made by replacing in the definitions the concept of small submodule by the corresponding one of $\delta$-small submodule. Most properties of lifting and supplemented modules are adapted to this new setting.

A submodule $N$ of $M$ is called $e$-small (essentially small) in $M$, denote $N \ll_e M$, if $M = N + L$ and $L \leq^e M$ then $L = M$ ([20]). In [12], the authors were introduced a class of all $e$-lifting modules. A module $M$ is called $e$-lifting if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll_e M$. Some homology properties of $e$-lifting modules class were obtained. It proved that $\text{Rad}_e(M)$ is a Noetherian (Artinian) module if only if $M$ has ACC(reps. DCC) on $e$-small submodules.

In [19], the author denoted 
\[
\delta(M) = \text{Rej}_M(\varnothing) = \bigcap \{N \leq M | M/N \in \varnothing\} = \sum \{N \leq M | N \ll_\delta M\}
\]
where $\varnothing$ is the class of all singular simple modules. Similarly, there is the concept of modules via $e$-small submodules ([20]). Call $\varnothing_0$ the class of all essential maximal submodules of $M$.

\[
\text{Rad}_e(M) = \bigcap \{N \leq M | N \in \varnothing_0\} = \sum \{N \leq M | N \ll_e M\}.
\]

Note that $\text{Rad}(M) \leq \delta(M) \leq \text{Rad}_e(M)$. If $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of $M$, $M$ is called a $\delta$-local module ([4]). In [15], the author studied $\delta$-local modules and established some properties of finitely generated amply $\delta$-supplemented modules. A necessary and sufficient condition is provided for a module to be $\delta$-local module. In this paper, we continue studying class of $e$-supplemented modules and introduce the concept of $e$-local modules. A module $M$ is called $e$-local if $\text{Rad}_e(M)$ is a maximal submodule of $M$ and $\text{Rad}_e(M) \ll_e M$. We show that $M = N \oplus K$ is an $e$-local module if and only if either $N$ is an $e$-local module and $K$ is semisimple, or $K$ is an $e$-local module and $N$ is semisimple.

Recall that the singular submodule of a module $M$ is the set
\[
Z(M) = \{m \in M | r(m) \leq^e R\}.
\]
In [6], the author introduced the notions of singular modules and nonsingular modules. A module $M$ is called singular (resp., nonsingular) if $Z(M) = M$ (resp., $Z(M) = 0$). In [13], the author defined the notion of dual singular submodules, that is $\overline{Z}(M) = \bigcap \{\text{Ker} \ g | g: M \to N, N$ is a small module}. $M$ is called cosingular (resp., noncosingular) module if $\overline{Z}(M) = 0$ (resp., $\overline{Z}(M) = M$). A generalization
of cosingular and noncosingular, which is \( \delta \)-cosingular and \( \delta \)-noncosingular (respectively) were introduced and studied in [10].

In [8], the authors introduce the notion of \( \mathcal{T} \)-noncosingular modules as the notion of dual \( \mathcal{K} \)-nonsingular modules and generalizations of noncosingular modules. It turns out that some results about \( \mathcal{K} \)-nonsingular modules hold for dual \( \mathcal{T} \)-noncosingular modules. The structure of finitely generated \( \mathcal{T} \)-noncosingular \( \mathbb{Z} \)-modules is described, and a necessary and sufficient condition is provided for a direct sum of \( \mathcal{T} \)-noncosingular modules to be \( \mathcal{T} \)-noncosingular. Rings for which all right modules are \( \mathcal{T} \)-noncosingular are shown to be precisely right V-rings. A module \( M \) is called \( \mathcal{T} \)-noncosingular relative to \( N \) if, for every nonzero homomorphism \( f : M \to N \), \( \text{Im} f \) is not small in \( N \). \( M \) is called \( \mathcal{T} \)-noncosingular if \( M \) is \( \mathcal{T} \)-noncosingular relative to \( M \).

In this paper, we introduce to a special case of \( \mathcal{T} \)-noncosingular modules which are \( \mathcal{T} \)-e-noncosingular modules. A module \( M \) is called \( \mathcal{T} \)-e-noncosingular relative to \( N \) if, for every nonzero homomorphism \( f : M \to N \), \( \text{Im} f \) is not e-small in \( N \). \( M \) is called \( \mathcal{T} \)-e-noncosingular if \( M \) is \( \mathcal{T} \)-e-noncosingular relative to \( M \). Some properties of this class of modules and the relation to other kinds of modules are shown in section 3. We show that every right \( R \)-module is \( \mathcal{T} \)-e-noncosingular if and only if every right \( R \)-module is e-noncosingular, if and only if for any right \( R \)-module \( M \), \( \text{Rad}_e(M) = 0 \). Furthermore, \( \mathcal{T} \)-e-noncosingular modules and e-lifting modules are dual Baer modules.

### 2. e-local Modules

Recall that a submodule \( N \) of \( M \) is said to be e-small in \( M \) (denoted by \( N \ll_e M \)), if \( N + L = M \) with \( L \leq_e M \) implies \( L = M \).

The following lemma is proved in [20]:

**Lemma 2.1.** Let \( M \) be a module. Then

1. If \( N \ll_e M \) and \( K \leq N \), then \( K \ll_e M \) and \( N/K \ll_e M/K \).

2. Let \( N \ll_e M \) and \( M = X + N \). Then \( M = X \oplus Y \) for some a semisimple submodule \( Y \) of \( M \).

3. Let \( N, K \leq M \). Then \( N + K \ll_e M \) if and only if \( N \ll_e M \) and \( K \ll_e M \).

4. If \( K \ll_e M \) and \( f : M \to N \) is a homomorphism, then \( f(K) \ll_e N \). In particular, if \( K \ll_e M \leq N \), then \( K \ll_e N \).

5. Let \( K_1 \leq M_1 \leq M, K_2 \leq M_2 \leq M \) and \( M = M_1 \oplus M_2 \). Then \( K_1 \oplus K_2 \) is e-small in \( M_1 \oplus M_2 \) if and only if \( K_1 \ll_e M_1 \) and \( K_2 \ll_e M_2 \).

**Lemma 2.2.** Let \( M \) be an \( R \)-module and \( x \in M \). The following conditions are equivalent:

1. \( x \in \text{Rad}_e(M) \);

2. \( xR \ll_e M \).
Proof. It is clear and omit the proof. \hfill \Box

**Corollary 2.3.** If $M = \bigoplus_{i \in I} M_i$, then $\operatorname{Rad}_e(M) = \bigoplus_{i \in I} \operatorname{Rad}_e(M_i)$.

**Proof.** It is clear $\bigoplus_{i \in I} \operatorname{Rad}_e(M_i) \leq \operatorname{Rad}_e(M)$. For every $j \in I$, call $\pi_j : M \to M_j$ the canonical projection. If $x \in \operatorname{Rad}_e(M)$, then $xR \ll_e M$. It follows that $\pi_j(xR) \ll_e M_j$ or $\pi_j(x) \in \operatorname{Rad}_e(M_j)$. This gives $x \in \bigoplus_{i \in I} \operatorname{Rad}_e(M_i)$. \hfill \Box

**Lemma 2.4.** Let $M$ be a module. The following are equivalent:

1. $M \ll_e M$;
2. $M$ is a semisimple module;
3. Any submodule of $M$ is $e$-small in $M$.

**Proof.** (1) $\Rightarrow$ (2). Let $A$ and $B$ be submodules of $M$ with $A \oplus B \leq_e M$. As $M = M + (A \oplus B)$ and $M \ll_e M$, then $M = A \oplus B$. It follows that $M$ is a semisimple module.

(2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) are obvious. \hfill \Box

Recall that a module $M$ is called local if the sum of all proper submodules of $M$ is also a proper submodule of $M$. We call $M$ an $e$-local module if $\operatorname{Rad}_e(M)$ is a maximal submodule of $M$ and $\operatorname{Rad}_e(M) \ll_e M$.

Let $N, L$ be submodules of $M$. $L$ is called an $e$-supplement of $N$ in $M$ if $M = N + L$ and $N \cap L$ is $e$-small in $L$. A module $M$ is called $e$-supplemented if every submodule of $M$ has an $e$-supplement in $M$ [12].

**Lemma 2.5.** Any $e$-local module is $e$-supplemented.

**Proof.** Let $M$ be an $e$-local module and $N$ be a proper submodule of $M$. Since $\operatorname{Rad}_e(M)$ is a maximal submodule of $M$, either $N \leq \operatorname{Rad}_e(M)$ or $\operatorname{Rad}_e(M) + N = M$. If $N \leq \operatorname{Rad}_e(M)$ then $M$ is an $e$-supplement of $N$ in $M$. Now suppose $N + \operatorname{Rad}_e(M) = M$. It follows that $N \oplus Y = M$ for some semisimple submodule $Y$ of $M$. Clearly, $Y$ is an $e$-supplement of $N$ in $M$. Thus $M$ is $e$-supplemented. \hfill \Box

**Remark 2.6.** The following statements hold

1. Every simple module is local.
2. Every semisimple module $M$ is not $e$-local, since $\operatorname{Rad}_e(M) = M$.

We next give some characterizations of $e$-local modules with semisimple property. Furthermore, the relationship between of $e$-local modules and local modules are considered.

**Proposition 2.7.** Every local module is either simple or $e$-local.

**Proof.** Assume that $L$ is a local module and not simple. It is well-known that $\operatorname{Rad}(L)$ is the unique maximal submodule of $L$, $\operatorname{Rad}(L) \ll L$ and $\operatorname{Rad}(L) \leq_e L$. 

Suppose that $\text{Rad}_e(L) \neq \text{Rad}(L)$. Call $x \in \text{Rad}_e(L)$ and $x \not\in \text{Rad}(L)$. Then $xR \ll_e L$ by Lemma 2.2. Since $xR + \text{Rad}(L) = L$ and $\text{Rad}(L) \ll L$, then we have $xR = L$. Hence, $L \ll_e L$. By Lemma 2.4, $L$ is semisimple. So, $\text{Rad}(L) = 0$. Let $H$ be a proper submodule of $M$. Since $\text{Rad}(L)$ is an only maximal submodule of $M$, $H \leq \text{Rad}(L)$. Hence, $H = 0$. It follows that $M$ is simple, a contradiction. Thus, $\text{Rad}_e(L) \leq \text{Rad}(L)$. On the other hand, since $\text{Rad}(L) \ll L$, we have $\text{Rad}(L) \leq \text{Rad}_e(L)$. Thus $\text{Rad}(L) = \text{Rad}_e(L)$ is a maximal submodule of $L$ and $e$-small in $L$.

\begin{proposition}
The following conditions are equivalent for an $e$-local module $M$:

1. $M$ is local;
2. $M$ is an indecomposable module.

\end{proposition}

\begin{proof}
(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Note that $\text{Rad}_e(M)$ is a maximal submodule of $M$. Let $L$ be a proper submodule of $M$. Suppose that $L \not\subseteq \text{Rad}_e(M)$. Then $L + \text{Rad}_e(M) = M$. Since $\text{Rad}_e(M) \ll_e M$, there is a decomposition $M = L \oplus L'$ with $L'$ semisimple. But $M$ is indecomposable. Thus $L = M$ or $L = 0$. But $L \not\subseteq \text{Rad}_e(M)$ and so $L = M$, a contradiction. It follows that $L \leq \text{Rad}_e(M)$. Consequently, $M$ is a local module.

\end{proof}

\begin{theorem}
Let $M = N \oplus K$ be a module. The following statements are equivalent:

1. $M$ is $e$-local;
2. Either (a) $N$ is $e$-local and $K$ is semisimple, or (b) $K$ is $e$-local and $N$ is semisimple.

\end{theorem}

\begin{proof}
By Corollary 2.3, we have $\text{Rad}_e(M) = \text{Rad}_e(N) \oplus \text{Rad}_e(K)$.

(1) $\Rightarrow$ (2). Since $\text{Rad}_e(M)$ is a maximal submodule of $M$, we have

$$\text{Rad}_e(N) = N \text{ or } \text{Rad}_e(K) = K.$$ 

Assume that $\text{Rad}_e(N) = N$. If $X$ is a submodule of $K$ with $\text{Rad}_e(K) \leq X$, then $\text{Rad}_e(M) \leq N \oplus X$. So $X = \text{Rad}_e(K)$ or $X = K$. Therefore $\text{Rad}_e(K)$ is a maximal submodule of $K$. Moreover, $\text{Rad}_e(K)$ is $e$-small in $K$ and $N \ll_e N$. Thus $K$ is $e$-local and $N$ is semisimple by Lemma 2.4.

Similarly, if $\text{Rad}_e(K) = K$, then we also have $N$ is $e$-local and $K$ is semisimple.

(2) $\Rightarrow$ (1). Assume that $K$ is $e$-local and $N$ is semisimple. Then $N \ll_e N$ and $\text{Rad}_e(N) = N$ by Lemma 2.4. So $\text{Rad}_e(M) = N \oplus \text{Rad}_e(K) \ll_e M$. Let $L \leq M$ be a submodule such that $\text{Rad}_e(M) \leq L$. It follows that $\text{Rad}_e(K) \leq K \cap L$. As $\text{Rad}_e(K)$ is a maximal submodule of $K$, we have $K \cap L = \text{Rad}_e(K)$ or $K \cap L = K$. Note that $L = N \oplus (K \cap L)$. This gives that $L = \text{Rad}_e(M)$ or $L = M$. Therefore $\text{Rad}_e(M)$ is a maximal submodule of $M$. Consequently, $M$ is an $e$-local module.

\end{proof}
Corollary 2.10. A direct sum of two e-local modules is never e-local.

Proof. Let $M = L_1 \oplus L_2$ be a module with e-local modules $L_1$ and $L_2$. Suppose that $M$ is e-local. By Theorem 2.9, one of the $L_i$ ($i = 1, 2$) is semisimple. It follows that $\text{Rad}_e(L_1) = L_1$ or $\text{Rad}_e(L_2) = L_2$, a contradiction.

Example 2.11.

(1) Let $M$ be a simple singular module. Then $M$ is $\delta$-local but it is not e-local. For example, $M = \mathbb{Z}/p\mathbb{Z}$, $p$ is a prime number. Then $M$ is a $\mathbb{Z}$-module simple and singular.

(2) Let $N$ be an e-local projective module and $K$, a non-projective semisimple module. By Theorem 2.9 and [15, Proposition 2.17], $N \oplus K$ is an e-local module but it is not $\delta$-local.

(3) Let $R = \mathbb{Z}, M = \mathbb{Z}/24\mathbb{Z}$. Then, $\text{Rad}(M) = \delta(M) = 6M, \text{Rad}_e(M) = 2M$. So, $M$ is an e-local module but it is neither local nor $\delta$-local.

(4) Let $F$ be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $R$ is $\delta$-local but it is not local ([15, 2.5]). Moreover, $R$ is an e-local module by projectivity of $R$.

Proposition 2.12. A module $M$ is e-local if and only if $M = L \oplus N$ such that $L$ is a cyclic e-local module and $N$ is a semisimple module.

Proof. ($\Rightarrow$). Assume that $M$ is an e-local module. Then $\text{Rad}_e(M)$ is a maximal submodule of $M$. Call $x \in M$ and $x \notin \text{Rad}_e(M)$. By maximality of $\text{Rad}_e(M)$, $M = \text{Rad}_e(M) + xR$. Furthermore, $\text{Rad}_e(M) \ll_e M$, there exists a nonzero semisimple submodule $X$ of $M$ such that $M = X \oplus xR$. It follows that $\text{Rad}_e(X) = X$ and so $X$ is not e-local. We deduce that $xR$ is e-local by Theorem 2.9.

($\Leftarrow$). By Theorem 2.9.

Theorem 2.13. The following conditions are equivalent for a module $M$:

1. $M$ is an e-local module;
2. $\text{Rad}_e(M)$ is a maximal submodule of $M$ and every proper essential submodule of $M$ is contained in a maximal submodule;
3. $M$ has a unique essential maximal submodule and every proper essential submodule of $M$ is contained in a maximal submodule.

Proof. (1) $\iff$ (2) is clear.

1 $\Rightarrow$ (3). Since $M$ is e-local, $M$ is not semisimple. Assume that there is a nonzero submodule $X \leq M$ such that $\text{Rad}_e(M) \cap X = 0$. Since $\text{Rad}_e(M)$ is a maximal submodule of $M$, $M = \text{Rad}_e(M) \oplus X$. This gives that $X$ is a simple module. As $\text{Rad}_e(M) \ll_e M$, there exists a semisimple submodule $L \leq M$ such that $M = L \oplus X$. We deduce that $M$ is a semisimple module, a contradiction. It follows that $\text{Rad}_e(M)$ is essential in $M$. Now suppose that $M$ contains an essential
maximal submodule $N$ such that $N \nsubseteq \text{Rad}_e(M)$. Then $M = \text{Rad}_e(M) + N$. Since $\text{Rad}_e(M) \ll_e M$, there exists a semisimple submodule $E$ of $M$ such that $M = E \oplus N$. But $N$ is essential in $M$, we have $E = 0$ and so $N = M$, a contradiction. Consequently, $\text{Rad}_e(M)$ is the only essential maximal submodule of $M$.

(3) $\Rightarrow$ (1). Assume that every proper essential submodule $M$ is contained in a maximal submodule and $K$ is the only essential maximal submodule of $M$. If $x \in M \setminus K$, then $M = xR + K$ by maximality of $K$. By our assumption $K \leq M$, $xR$ is not $e$-small in $M$. This gives that $x \notin \text{Rad}_e(M)$. We deduce that $\text{Rad}_e(M) \leq K$. Let $Y$ be a proper essential submodule $M$, then $Y \leq K$ and $Y + K = K \neq M$. It follows that $K \ll_e M$, i.e. $K \leq \text{Rad}_e(M)$. Thus $\text{Rad}_e(M) = K \ll_e M$. \hfill $\square$

Following [12], a module $M$ is called $e$-supplemented if every submodule of $M$ has an $e$-supplement in $M$. A module $M$ is called amply $e$-supplemented if for any submodules $A, B$ of $M$ with $M = A + B$, there exists an $e$-supplement $P$ of $A$ such that $P \leq B$. In this case, we say that $A$ has ample $e$-supplements in $M$.

**Proposition 2.14.** Let $M$ be an $e$-local module. If $N$ is a submodule of $M$, then $N$ is either $e$-small in $M$ or there exists a semisimple submodule $X$ of $M$ such that $M = N \oplus X$.

**Proof.** Let $N$ be a submodule of $M$. Assume $N$ is not $e$-small in $M$. Then $N \nsubseteq \text{Rad}_e(M)$. By maximality of $\text{Rad}_e(M)$, we have $N + \text{Rad}_e(M) = M$. As $\text{Rad}_e(M) \ll_e M$, $M = N \oplus X$ for some a semisimple submodule $X$ of $M$. \hfill $\square$

**Lemma 2.15.** Let $N$ be a maximal submodule of a module $M$. If $K$ is an $e$-supplement of $N$ in $M$, then $K$ is either $e$-local or semisimple.

**Proof.** By assumption, we have $N + K = M$ and $N \cap K \ll_e K$. Therefore $N \cap K \leq \text{Rad}_e(K)$. As $M/N \simeq K/(N \cap K)$, $N \cap K$ is a maximal submodule of $K$. It follows that $\text{Rad}_e(K) = N \cap K$ or $\text{Rad}_e(K) = K$. If $\text{Rad}_e(K) = N \cap K$, then $K$ is an $e$-local module. Assume that $\text{Rad}_e(K) = K$. For any $x \in K \setminus (N \cap K)$, we have $xR + (N \cap K) = K$. Furthermore, we have $xR \ll_e K$ by Lemma 2.2 and $N \cap K \ll_e K$. Thus $K \ll_e K$ by Lemma 2.1. By Lemma 2.4, $K$ is a semisimple module. \hfill $\square$

**Lemma 2.16.** Let $L_1, L_2, \ldots, L_n$ be submodules of $M$ such that either $L_i$ is $e$-local or $L_i$ is semisimple. Assume that $N$ is a submodule of $M$ and $N + L_1 + \ldots + L_n$ has an $e$-supplement $K$ in $M$. Then, there exists a subset $I$ of $\{1, \ldots, n\}$ such that $K + \sum_{i \in I} X_i$ is an $e$-supplement of $N$ in $M$, where $X_i = L_i$ or $X_i$ is a semisimple direct summand of $L_i$.

**Proof.** If $n = 1$ then $N + (K + L_1) = M$ and $K \cap (N + L_1) \ll_e K$. Call $H = (N + K) \cap L_1$. Assume that $H \ll_e L_1$. We have

$$N \cap (K + L_1) \leq [(N + L_1) \cap K] + [(N + K) \cap L_1 \ll_e K + L_1].$$

It follows that $K + L_1$ is an $e$-supplement of $N$ in $M$. 

If \( H \not\leq_e L_1 \) then \( L_1 \) is not semisimple by Lemma 2.4. By hypothesis, \( L_1 \) is \( e \)-local. From Proposition 2.14, there exists a semisimple submodule \( X_1 \leq L_1 \) such that \( H \oplus X_1 = L_1 \). Hence \( N + (K + X_1) = M \). We have that
\[
N \cap (K + X_1) \leq (N + K) \cap X_1 + (N + X_1) \cap K,
\]
and obtain that \( N \cap (K + X_1) \leq_e K + X_1 \). This gives that \( K + X_1 \) is an \( e \)-supplement of \( N \) in \( M \).

Proposition 2.17. Let \( M \) be a finitely generated module. The following conditions are equivalent:

1. \( M \) is an amply \( e \)-supplemented module;
2. Every maximal submodule of \( M \) has ample \( e \)-supplement in \( M \);
3. If \( L, N \) are submodules of \( M \) and \( M = L + N \) then \( M = N + L_1 + \ldots + L_n \), where \( n \) is a positive integer number, either \( L_i \) is \( e \)-local or \( L_i \) is semisimple.

Proof. (1) \( \Rightarrow \) (2). It is clear.

(2) \( \Rightarrow \) (3). Let \( N, L \) be submodules of \( M \) and \( M = N + L \). Call \( \Gamma \) a class of all submodules \( X \) of \( M \) such that \( X \leq L \) and \( X = X_1 + \ldots + X_k \), where either \( X_i \) is \( e \)-local or \( X_i \) is semisimple. Assume that \( M \neq N + A \) for all \( A \in \Gamma \). By [15, Lemma 35], there exists a submodule \( U \leq M \) such that \( N \leq U \) and \( U \) is a maximal submodule of \( M \) satisfying \( M \neq U + A \) for all \( A \in \Gamma \). Since \( M \) is finitely generated and \( U \neq M \), there exists a maximal submodule \( K \leq M \) such that \( U \leq K \). So \( K + L = M \). By hypothesis, there exists a submodule \( E \leq L \) such that \( E \) is an \( e \)-supplement of \( K \) in \( M \). Following Lemma 2.15, either \( E \) is \( e \)-local or \( E \) is semisimple. It is easy to see that \( U \neq U + E \). Otherwise, we have \( E \leq U \leq K \) and \( K = K + E = M \). It follows \( M = U + E + F, F \in \Gamma \). So \( E + F \in \Gamma \), a contradiction.

(3) \( \Rightarrow \) (1). By Lemma 2.16.

Lemma 2.18. Let \( N, L \) be submodules of \( M \) such that \( M = N + L \). If \( L \) is an \( e \)-supplemented module then \( L \) contains an \( e \)-supplement of \( N \) in \( M \).

Proof. By hypothesis, there exists a submodule \( K \) of \( L \) such that \( (N \cap L) + K = L \) and \( (N \cap L) \cap K \leq_e K \). Then \( N + K = M \) and \( N \cap K \leq_e K \). So \( K \) is an \( e \)-supplement of \( N \) in \( M \).

Proposition 2.19. Let \( M \) be a module. If every cyclic submodule of \( M \) is \( e \)-supplemented then every maximal submodule of \( M \) has ample \( e \)-supplement.
Proof. Assume that $N$ is a maximal submodule of $M$. Let $L$ be a submodule of $M$ such that $M = N + L$. There exists $x$ in $L$ satisfying $x \notin N$ and $xR + N = M$. Following Lemma 2.18, $xR$ contain an $e$-supplement of $N$ in $M$. \qed

Corollary 2.20. If $M$ is a finitely generated module and every cyclic submodule of $M$ is $e$-supplemented then $M$ is an $e$-supplemented module.

Proof. By Proposition 2.17 and Proposition 2.19. \qed

3. $T$-e-noncosingular Modules

Let $M, N$ be right $R$-modules. We call $M$ $T$-e-noncosingular relative to $N$ if $\text{Im} f$ is not $e$-small in $N$ for any nonzero homomorphism $f : M \to N$. $M$ is called $T$-e-noncosingular if $M$ is $T$-e-noncosingular relative to $M$. The ring $R$ is called right (left) $T$-e-noncosingular if the right (left) module $R$ is $T$-e-noncosingular, respectively.

We denote $\nabla_e[M, N] = \{f : M \to N | \text{Im } f \ll_e N\}$. It is easily to check that $M$ is $T$-e-noncosingular relative to $N$ if and only if $\nabla_e[M, N] = \emptyset$.

Proposition 3.1. Let $M, N$ be right $R$-modules and $K$ is a direct summand of $M$. If $\nabla_e[M, N] = 0$ then $\nabla_e[K, N] = 0$.

Proof. Assume that $M = K \oplus L$ and $\varphi \in \nabla_e[K, N]$. Then $\text{Im } \varphi \ll_e N$. We consider the homomorphism $\varphi \oplus 0_L : M \to N$ defined by $(\varphi \oplus 0_L)(k + l) = \varphi(k)$ for all $k \in K, l \in L$. So $\text{Im}(\varphi \oplus 0_L) = \text{Im } \varphi \ll_e N$. As $\nabla_e[M, N] = 0$, $\varphi \oplus 0_K = 0$ and hence $\varphi = 0$. \qed

Proposition 3.2. Let $M, N$ be right $R$-modules. If $\nabla_e[M, N] = 0$ then $\nabla_e[M, P] = 0$ for all submodule $P$ of $N$.

Proof. Assume that $P \subseteq N$ and $\varphi \in \nabla_e[M, P]$. Then $\text{Im } \varphi \ll_e P$. It follows that $\text{Im } \varphi \ll_e N$. Since $\nabla_e[M, N] = 0$, $\varphi = 0$. \qed

Corollary 3.3. Every direct summand of a $T$-e-noncosingular module is also a $T$-e-noncosingular module.

Proof. It is followed from Proposition 3.1. \qed

Proposition 3.4. Let $M = \oplus_{i \in I} M_i$, $N = \oplus_{j \in J} N_j$ be right $R$-modules, where $I, J$ are non-empty sets. Then $\nabla_e[M, N] = 0$ if only if $\nabla_e[M_i, N_j] = 0$ for all $i \in I, j \in J$.

Proof. Assume that $\nabla_e[M_i, N_j] = 0$ for all $i \in I, j \in J$. Let $f \in \nabla_e[M, N]$ and the conclusion $f_i : M_i \to M$. Since $\text{Im } f \ll_e N_j$, $\text{Im } f_i \ll_e N_j$ for all $i \in I$. Hence $f_i = 0$ for all $i \in I$. It follows that $f = 0$. Now, let $\varphi \in \nabla_e[M, N]$ and the projection $\pi_j : N \to N_j$. Set $\varphi_j = \pi_j \varphi : M \to N_j$. Since $\text{Im } \varphi \ll_e N$, $\text{Im } \varphi_j \ll_e N_j$ for all $i \in I$. By hypothesis, $\varphi_j = 0$. It follows that $\varphi = 0$. \qed
Corollary 3.5. Let $M = \oplus_{i \in I} M_i$, $N = \oplus_{j \in J} N_j$ be right $R$-modules, where $I, J$ are non-empty sets. Then $M$ is $T$-e-noncosingular relative to $N$ if only if $M_i$ is $T$-e-noncosingular relative to $N_j$ for all $i \in I, j \in J$.

Corollary 3.6. Let $(M_i)_{i \in I}$ be a family of modules. Then $M = \oplus_{i \in I} M_i$ is a $T$-e-noncosingular if and only if $M_i$ is $T$-e-noncosingular relative to $M_j$ for all $i, j \in I$.

Let $M$ be a module. We call $M$ an $e$-small module if $M$ is $e$-small in injective envelope of $M$. We denote $Z_e(M) = \bigcap \{ \ker g : M \to N, N$ is $e$-small module $\}$.

If $Z_e(M) = M$, then $M$ is called an $e$-noncosingular module.

Proposition 3.7. The following conditions are equivalent for a ring $R$:

1. Every right $R$-module is $T$-e-noncosingular;
2. Every right $R$-module is $e$-noncosingular;
3. For any right $R$-module $M$, $\text{Rad}_e(M) = 0$.

Proof. (1) \Rightarrow (2). Let $N \ll_e E(N)$. We will prove $N = 0$. We consider the homomorphism $f : M \oplus N \to E(N)$ given by $f(m + n) = n$ for all $m \in M, n \in N$. Then $\text{Im } f = N \ll_e E(N)$. We have that $M \oplus N \oplus E(N)$ is an $T$-e-noncosingular module and obtain that $M \oplus N$ is $T$-e-noncosingular relative to $E(N)$. This gives $f = 0$. It is easily to check that $N = 0$. Furthermore, for any $R$-module $M$, $Z_e(M) = \bigcap \{ \ker g : M \to 0 \} = M$, i.e., $M$ is $e$-noncosingular.

(2) \Rightarrow (3). Assume that $N$ is an $e$-small submodule of $M$. Call $\pi : M \oplus N \to N$ the projection. By hypothesis, $M \oplus N$ is $e$-noncosingular. We have that $Z_e(M \oplus N) = M \oplus N$ and obtain that $f = 0$. Thus $N = 0$.

(3) \Rightarrow (1). It is clear. \qed

Now, we denote:

$Z_{e-M}(N) = \bigcap_{\varphi \in \nabla_e[M,N]} \ker \varphi$

Proposition 3.8. Let $M$ be a module. Then the following conditions hold:

1. $Z_e(M) \leq Z_{e-M}(N)$.
2. $Z_{e-M}(N)$ is a fully invariant submodule of $M$.
3. $\nabla_e[M,N] = 0$ if only if $M = Z_{e-M}(N)$.
4. If $M = \oplus_{i \in I} M_i$ then $Z_{e-M}(N) \leq \oplus_{i \in I} Z_{e-M_i}(N)$.
Proof. (1) By definition, we get
\[ \mathcal{Z}_e(M) \leq \bigcap \{ \text{Ker} g : M \to N | N = \text{Im } f, f \in \nabla_e[M, N] \} = \mathcal{Z}_{e-M}(N). \]

(2) Assume \( f \in \text{End}(M) \) and \( \varphi \in \text{Hom}(M, N) \) such that \( \text{Im } \varphi \leq_e N \). Therefore \( \text{Im } \varphi f \leq \text{Im } \varphi \). So \( \text{Im } \varphi f \leq_e N \). For all \( x \in \mathcal{Z}_{e-M}(N) \), \( \varphi(x) = 0 \) implies \( \varphi f(x) = 0 \). Thus \( f(x) \in \mathcal{Z}_{e-M}(N) \), i.e., \( \mathcal{Z}_{e-M}(N) \) is fully invariant.

(3) It is clear.

(4) As \( \mathcal{Z}_{e-M}(N) \) is fully invariant, \( \mathcal{Z}_{e-M}(N) = \oplus_{i \in I}(\mathcal{Z}_{e-M}(N) \cap M_i). \) We will prove that \( \mathcal{Z}_{e-M}(N) \cap M_i \subset \mathcal{Z}_{e-M}(N) \). Let \( x_i \in \mathcal{Z}_{e-M}(N) \cap M_i \) and \( \varphi_i : M_i \to N \) such that \( \text{Im } \varphi_i \leq_e N \). Then \( \psi_i : M \to M \) extends \( \varphi_i \) (\( \psi_i |_{M_j} = 0 \) for all \( j \neq i \)). This gives \( \text{Im } \psi_i \leq_e N \). Thus \( \psi_i(x_i) = 0 \) and hence \( x_i \in \mathcal{Z}_{e-M}(N) \).

Corollary 3.9. Let \( M \) and \( N \) be modules. Then \( M \) is \( \mathcal{T}-e \)-noncosingular relative to \( N \) if and only if \( \mathcal{Z}_{e-M}(N) = M \).

Remark 3.10. It is clearly to see that \( \mathcal{Z}_{e-M}(M) \leq \mathcal{Z}_{\mathcal{T}}(M) \leq \bigcap \{ \text{Ker} \varphi | \varphi \in \text{End}(M), \text{Im } \varphi \leq M \}. \) So, if \( M \) is a \( \mathcal{T}-e \)-noncosingular then \( M \) is a \( \mathcal{T} \)-noncosingular module. The converse is not true in general.

Example 3.11.

(1) \( \mathbb{Z} \)-module \( \mathbb{Z} \) is \( \mathcal{T}-e \)-noncosingular.

(2) If \( M_\mathbb{Z} = \mathbb{Z}_6 \) then \( \text{Rad}(M) = 0 \) and \( \mathcal{Z}_{e-M}(M) = 0 \). It follows that \( M \) is \( \mathcal{T} \)-noncosingular but not \( \mathcal{T}-e \)-noncosingular.

(3) Let \( R \) be a proper Dedekind domain and \( P \) be a nonzero prime ideal of \( R \). Consider module \( M = R(P^\infty) \oplus R/P \). Then \( M \) is not a \( \mathcal{T} \)-noncosingular module (see Example 2.12, [9]). So \( M \) is not a \( \mathcal{T}-e \)-noncosingular module.

(4) As \( \text{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z}_2) = \text{Hom}_\mathbb{Z}(\mathbb{Z}_2, \mathbb{Q}) = 0 \), \( \mathbb{Q}_\mathbb{Z} \) is \( \mathcal{T}-e \)-noncosingular relative to \( \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \) is \( \mathcal{T}-e \)-noncosingular relative to \( \mathbb{Q} \). Hence \( (\mathbb{Q} \oplus \mathbb{Z}_2)_\mathbb{Z} \) is \( \mathcal{T}-e \)-noncosingular by Lemma 3.6.

Proposition 3.12. Let \( M \) be an \( R \)-module which \( S = \text{End}(M) \) is Von Neumann regular and \( T(M) = \{ N \leq M | \text{Rad}_e(N) = N \} \). If \( T(M) = 0 \) then \( M \) is \( \mathcal{T}-e \)-noncosingular.

Proof. Let \( f \in \text{End}(M) \) such that \( \text{Im } f \leq_e M \). Then \( \text{Im } f \leq \text{Rad}_e(M) \). Since \( S \) is regular, there exists \( g \in S \) such that \( f = fgf \). Hence \( fg \) is an idempotent and \( M = \text{Im } fg \oplus \text{Ker } fg \). Since \( \text{Im } fg \leq \text{Im } f \leq \text{Rad}_e(M) \), \( \text{Rad}_e(M) = \text{Rad}_e(\text{Im } fg) \oplus \text{Rad}_e(\text{Ker } fg) \). So, \( \text{Im } fg \cap \text{Rad}_e(M) = \text{Im } fg = \text{Rad}_e(\text{Im } fg) \oplus (\text{Im } fg \cap \text{Rad}_e(\text{Ker } fg)) \). It follows \( \text{Im } fg = \text{Rad}_e(\text{Im } fg) \). Therefore \( \text{Im } fg \in T(M) \). We have \( fg = 0 \) and \( f = 0 \). \( \square \)
3.11. Then $M$ is not a $T$-noncosingular module. But the converse is not true in general. For example, let $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_2$ in Example 3.11. Then $M$ is $T$-noncosingular. However, we have

$$\text{Rad}_e(\mathbb{Q} \oplus \mathbb{Z}_2) = \text{Rad}_e(\mathbb{Q}) \oplus \text{Rad}_e(\mathbb{Z}_2) = 0 \oplus \mathbb{Z}_2 \neq 0.$$ 

**Proposition 3.13.** Let $M = xR$ be a cyclic module such that $r(x)$ is an ideal of $R$. Then $M$ is $T$-noncosingular if and only if $\text{Rad}_e(M) = 0$.

**Proof.** Assume that $M$ is $T$-noncosingular and $\text{Rad}_e(M) \neq 0$. There exists $a \in R$ such that $xa \neq 0$ and $xa \in \text{Rad}_e(M)$. Call $f$ an endomorphism of $M$ with $f(xr) = xar$ for all $r \in R$. We have $\text{Im} f \leq \text{Rad}_e(M)$ and $f \neq 0$. But $\text{Rad}_e(M) \ll_e M$, a contradiction. The converse is clear. \qed

**Corollary 3.14.** A ring $R$ is right $T$-noncosingular if and only if $\text{Rad}_e(R_R) = 0$.

**Example 3.15.**

1. Consider $\mathbb{Z}_6$ as a ring. We have $J(\mathbb{Z}_6) = 0$, $\text{Rad}_e(\mathbb{Z}_6) = \mathbb{Z}_6$. So $\mathbb{Z}_6$ is $T$-noncosingular.

2. Let $R$ be a discrete valuation ring with maximal ideal $m$. Then $R$ is not $T$-noncosingular following Example 4.7,[14]. So $R$ is not $T$-noncosingular.

For $N \leq M, I \leq S = \text{End}(M)$, denote $N \leq M$ means that $N$ is a fully invariant submodule of $M$ and $E_M(I) = \sum_{\phi \in I} \text{Im} \phi; D_S(N) = \{ \phi \mid \text{Im} \phi \leq N \}$.

**Lemma 3.16.** Let $N \leq M, I \leq S, P \leq M, L \leq S$. Then:

1. $E_M(D_S(E_M(I))) = E_M(I)$;
2. $D_S(E_M(D_S(N))) = D_S(N)$;
3. $E_M(L) \leq M$;
4. $D_S(P) \leq S$.

**Proof.**

1. $E_M(D_S(E_M(I))) = \sum_{\phi \in D_S(E_M(I))} \text{Im} \phi \leq E_M(I)$. Conversely, for all $\varphi \in I, \text{Im} \varphi \leq E_I(M)$. So $\varphi \in D_S(E_M(I)) = \{ \phi \mid \text{Im} \phi \leq E_M(I) \}$.

2. $E_M(D_S(N)) \leq N$ implies $D_S(E_M(D_S(N))) \leq D_S(N)$. Conversely, for all $\varphi, \text{Im} \varphi \leq N, \text{Im} \varphi \leq \sum_{\text{Im} \varphi \leq N} \text{Im} \phi = E_M(D_S(N))$. So $D_S(N) \leq D_S(E_M(D_S(N)))$.

3. Let $f : M \to M, f(E_M(L)) = \sum_{\phi \in L} f(\text{Im} \phi) = \sum_{\phi \in L} \text{Im} \phi f$. Since $L \leq S, \phi f \in L$.

So $\sum_{\phi \in L} \text{Im} \phi f \leq \sum_{\psi \in L} \text{Im} \psi = E_M(L)$.

4. For all $\psi \in S, \phi \in D_S(P)$. We have $\psi \phi(M) \leq \psi(P) \leq P$ and $\phi \psi(M) \leq \phi(M) \leq P$. So $\psi \phi \in D_S(P)$ and $\phi \psi \in D_S(P)$. \qed
Proposition 3.17. Let $M$ be an $R$-module. $M$ is $\mathcal{T}$-$e$-nonsingular if and only if for all $I \leq S, E_M(I) = eM \oplus L$, in which $L \ll_e M, e^2 = e \in S$ implies $I \cap (1-e)S = 0$.

Proof. ($\Rightarrow$). Assume $I \leq S, E_M(I) = eM \oplus L$, in which $L \ll_e M, e^2 = e \in S$. We have $E_M(I \cap (1-e)S) \leq E_M(I) \cap E_M((1-e)S) \leq E_M(I) \cap (1-e)M = (eM \oplus L) \cap (1-e)M \leq (1-e)L$. Since $L \ll_e M, (1-e)L \ll_e M$. Hence $E_M(I \cap (1-e)S) \ll_e M$. $M$ is $\mathcal{T}$-$e$-nonsingular, so $I \cap (1-e)S = 0$.

($\Leftarrow$). Let $\phi \in S, \text{Im} \phi \ll_e M$. We have $E_M(\phi S) = \sum_{\psi \in S} \text{Im} \psi = \phi \sum_{\psi \in S} \text{Im} \psi = \phi(M) \ll_e M$. By hypothesis, $I \cap S = 0$. Hence $I = 0$, i.e., $\phi = 0$. \hfill $\square$

Corollary 3.18. $M$ is a $\mathcal{T}$-$e$-nonsingular module if and only if for all $I \leq S, E_M(I) \ll_e M$ implies that $I = 0$.

Now, we call $M$ an $e-\mathcal{K}$-module if for all $N \leq M, D_S(N) = 0$ implies $N \ll_e M$.

Proposition 3.19. $M$ is an $e-\mathcal{K}$-module if and only if, for all $N \leq M, E_M(D_S(N))$ is a direct summand of $M$ implies that $N = E_M(D_S(N)) \oplus L$ with $L \ll_e M$.

Proof. Assume that $N \leq M$ and $E_M(D_S(N)) \leq^\oplus M$. Then $E_M(D_S(N)) = eM, e^2 = e \in S$. Clearly, $eM = E_M(D_S(N)) \leq N$. On the other hand, $D_S(eM) \cap D_S((1-e)M \cap N) = 0$ and $D_S((1-e)M \cap N) \leq D_S(N) = D_S(eM)$. Hence $D_S((1-e)M \cap N) = 0$. Since $M$ is an $e-\mathcal{K}$-module, we have $(1-e)M \cap N \ll_e M$. Thus $N = E_M(D_S(N)) \oplus ((1-e)M \cap N)$ and $(1-e)M \cap N \ll_e M$.

Conversely, assume $N \leq M$ and $D_S(N) = 0$. Then $E_M(D_S(N)) = 0$. By hypothesis, $N = E_M(D_S(N)) \oplus L$ with $L \ll_e M$. Thus $N = L \ll_e M$. \hfill $\square$

Recalled that a module $M$ is $e$-lifting if for all submodule $N$ of $M$, there exists decompsiton $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll_e B$ ([12]). A module $M$ is called dual Baer if for all $N \leq M$, there exists an idempotent $e \in S = \text{End}(M)$ such that $D_S(N) = eS([8])$.

Lemma 3.20. A dual Baer $e-\mathcal{K}$-module is $e$-lifting.

Proof. Assume $M$ is a dual Baer and $e-\mathcal{K}$-module. Let $N$ be a submodule of $M$. There exists an idempotent $e \in S = \text{End}(M)$ such that $D_S(N) = eS$. We have $eM = E_M(eS) \leq N$. Hence $N = eM \oplus ((1-e)M \cap N)$. For all $\phi \in D_S((1-e)M \cap N)$, $\text{Im} \phi \leq N$. It follows $\phi \in D_S(N) = eS$. Since $\phi(M) \leq (1-e)M \cap eM = 0$, then $\phi = 0$, i.e., $D_S((1-e)M \cap N) = 0$. Since $M$ is an $e-\mathcal{K}$-module, $(1-e)M \cap N \ll_e M$. Thus $M$ is $e$-lifting. \hfill $\square$

Theorem 3.21. A $\mathcal{T}$-$e$-nonsingular $e$-lifting module is dual Baer.

Proof. Assume that $M$ is a $\mathcal{T}$-$e$-nonsingular $e$-lifting module and $N \leq M$. Then $N = eM \oplus B$ which $e^2 = E \in S, B = (1-e)M \cap N \ll_e M$. Hence $eS \leq D_S(eM) \leq D_S(N)$. If there exists $\phi \in D_S(N) \setminus eS$, then $(1-e)\phi = eS \cap D_S(N)$. We obtain that $(1-e)\phi M \leq N$ and $(1-e)\phi M \leq (1-e)M$. So $(1-e)\phi M \leq N \cap (1-e)M = B \ll_e M$.
Since \( M \) is \( \mathcal{T}\)-noncosingular, which follows \((1 - e)\phi = 0\), i.e., \( \phi = e\phi \in eS \). This is a contradiction. Thus \( D_S(N) = eS \), i.e., \( M \) is dual Baer.

**Lemma 3.22.** Let \( M \) be a \( \mathcal{T}\)-noncosingular module and \( X \), a fully invariant submodule of \( M \) and \( X = N \oplus B \) with \( B \ll e M \). If \( N \) is a direct summand of \( M \) then \( N \) is a fully invariant submodule of \( M \).

**Proof.** Assume \( M = N \oplus P \) and \( \phi \in \text{End}(M) \). Set \( \psi = \pi_P\phi|_N \pi_N \). If there exists \( x \in N \) such that \( \phi(x) \notin N \), then \( \psi(x) 
eq 0 \). Since \( X \) is a fully invariant submodule of \( M \), \( \phi(N) \leq \phi(X) \leq X \). So

\[
\phi(M) = \pi_P\phi|_N \pi_N(M) = \pi_P\phi|_N(N) \leq \pi_P(X) = X \cap P.
\]

Then \( X \cap P \cong B \). It follows \( X \cap P \ll e M \). As \( M \) is \( \mathcal{T}\)-noncosingular, \( \psi = 0 \), a contradiction. Thus \( \phi(N) \leq N \). \( \square \)

**Proposition 3.23.** Let \( M \) be a \( \mathcal{T}\)-noncosingular module. The following conditions are equivalent:

1. For every fully invariant submodule \( N \) of \( M \), there exists a direct summand \( B \) of \( M \) such that \( N/B \ll e M/B \);
2. For every fully invariant submodule \( N \) of \( M \), there exists a fully invariant direct summand \( B \) of \( M \) such that \( N/B \ll e M/B \).

**Proof.** (2) \( \Rightarrow \) (1) is clear. It suffices to prove (1) \( \Rightarrow \) (2). Assume \( X \leq M \). By (1), we have \( X = N \oplus B \), \( B \ll e M \) and \( N \) is a direct summand of \( M \). By Lemma 3.22, \( N \) is a fully invariant submodule of \( M \). Thus (2) holds. \( \square \)

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**References**


