Critical Phenomena of the Generalized Epidemic Process

Meesoon Ha*

Department of Physics Education, Chosun University, Gwangju 61452, Korea

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We investigate mixed-order critical phenomena of the generalized epidemic process (GEP) on random scale-free networks characterized by the power-law distribution $p_k \sim k^{-\alpha}$. The GEP is a minimal model of spreading behaviors. Near tricritical points (TCPs) derived by using the generating method, we numerically confirm the associated scaling exponents as functions of $\alpha$. In particular, we propose an extended finite-size scaling theory of the GEP and crossover scaling behaviors, which are also confirmed by using extensive Monte Carlo simulations. Our results show that near TCPs, the GEP is governed by two distinct length scales, whose nontrivial dependence on $\alpha$ leads to rich transition behaviors.

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I. introduction

Ranging from behavior adoption [1] to co-infection of multiple diseases [2], spreading phenomena are complex contagions, where multiple cooperating spreaders try to convert a common neighbor enhances or reduces the probability of success. These typically occur in systems with heterogeneous structure, scale-free networks characterizing by a power-law degree distribution with the probability that an arbitrary node is linked to $k$ neighbors, $p_k \sim k^{-\alpha}$ [3]. Cooperative effects have been modeled in various ways. So far the bootstrap and $k$-core percolations [4,5] have received much attention as paradigmatic models describing the formation of a large-scale, highly-interconnected structure. The spreading dynamics has been more explicitly studied in terms of the Watts’ threshold model [6], the generalized epidemic process (GEP) [7] and references therein, cooperative co-epidemics [8,9], two-step contagion model [10], and many others. All of these models show that, as contagion becomes more infectious, a singular transition occurs from a contagion-localized phase to an active phase where the contagion spreads to a finite fraction of the entire population. Moreover, as the strength of cooperation is increased, the type of the phase transition changes from continuous (the bond percolation) to discontinuous. The marginal case exhibits a continuous transition with two critical scaling fields. The GEP, similar to cooperative directed percolation [11,12], illustrates nonequilibrium analogs of equilibrium tricritical points (TCPs) [13,14].

In this paper, we revisit mixed-order transitions of the GEP to discuss its phase diagrams with crossover scaling and the competition of relevant length scales. Based on the most recent work [7] analytically located TCPs, we obtain the two relevant length scales as functions of the power-law exponent $\alpha$. As a result, we propose an extended finite-size scaling (FSS) theory of the GEP, incorporating both dynamic and static aspects of tricritical phenomena. It is notable that for strong heterogeneity ($2<\alpha<3$). The results also indicate a nonzero fraction of infected nodes even in the limit of vanishing infection probability, reported in Ref. [7].

This paper is organized as follows. We describe the GEP and show its phase diagram where the TCPs are drawn as a function of $\alpha$ in Sec. II, where the network basic properties are discussed as well. In Sec. III, the extended FSS theory is summarized with the conjecture

*E-mail: msha@chosun.ac.kr
of two relevant length scales and crossover scaling behaviors, which is numerically confirmed. We conclude this paper with summary and some remarks in Sec. IV.

II. Model

The GEP on networks is implemented as follows. At every instant, a node is in one of the four states: unexposed (S1), exposed (S2), infected (I), and removed (R). Initially only a single node (the “seed”) is in the I-state, while the others are all in the S1-state. At every time step, a randomly chosen I-node attempts to infect all of its S1- or S2-neighbors, each of the former (latter) with an independent and identical probability $\lambda$ ($\mu$). The neighbors which have been successfully infected turn into I-nodes, and the S1-neighbors which remain uninfected become S2-nodes. After then the I-node immediately deactivates by changing its state permanently to R, no longer participating in the process. The same procedure is repeated until the network runs out of I-nodes. In the special case $\mu = \lambda$, the process is equivalent to the susceptible-infected-removed (SIR) model [15], a dynamic version of the bond percolation model. If the network consists of $N$ nodes and in the end there are $R$ nodes in the R-state, the fraction $r \equiv R/N$ is chosen to be the order parameter, so that a phase transition occurs when $r$ changes from zero to a positive value as $\lambda$ is varied. The transition point $\lambda = \lambda_c$ is also called the epidemic threshold.

Tree-like description of the GEP. — An ensemble of random scale-free networks is generated by a degree distribution $p_k = k^{-\alpha}/\zeta_\alpha$, where the degree $k$ of each node can be any integer not less than the minimum degree $k_m$, and the Hurwitz zeta function $\zeta_s \equiv \sum_{k=k_m}^\infty k^{-s}$ gives the normalization constant. The degrees of adjacent nodes are assumed to be uncorrelated. Moreover, the power-law exponent $\alpha$ is assumed to be larger than 2, so that the average degree $\langle k \rangle = \zeta_{\alpha-1}/\zeta_\alpha$ is a finite number, which are locally tree-like, in the sense that they rarely contain loops made of a finite number of links [16]. A network with this property can be decomposed into separate layers of nodes, each layer index $n$ indicating the distance from the seed.

In order to keep track of the spreading, we consider the probability $q_l$ that a node in the $l$-th layer, reached by following a randomly chosen link, ends up in the R-state. Utilizing the tree-like property, the probability $1-q_{l+1}$ of a node in the $(l+1)$-th layer being uninfected is related to the corresponding probability in the $l$-th layer by a recursive relation

$$1 - q_{l+1} = \sum_{k=k_m}^\infty p_k \left( 1 - q_l \right)^{k-1}$$

$$+ \sum_{n=1}^{k-1} \binom{k-1}{n} (1-\lambda)(1-\mu)^{n-1} q_l^n (1-q_l)^{k-1-n}, \quad (1)$$

where $p_k \equiv k^{\alpha}/\langle k \rangle$ is the degree distribution, weighted by $k$ because higher-degree nodes are more likely to be chosen. The first term in the summand represents the case when none of the neighbors in the $l$-th layer are infected, and the second term corresponds to the case when all of the $n$ infected neighbors fail to pass on contagion. Calculating the sums, Eq. (1) leads to a discrete map

$$q_{l+1} = f(q_l). \quad (2)$$

If $q_\infty = \lim_{l \to \infty} q_l$, the order parameter $r$ satisfies

$$1 - r = \sum_{k=k_m}^\infty p_k \left( 1 - q_\infty \right)^k$$

$$+ \sum_{n=1}^{k-1} \binom{k}{n} (1-\lambda)(1-\mu)^{n-1} q_\infty^n (1-q_\infty)^{k-n}. \quad (3)$$

For $q_\infty \ll 1$, the above equation implies

$$r = \langle k \rangle \lambda q_\infty + O\left( q_\infty^{\min[2,\alpha-1]} \right). \quad (4)$$

Thus the transition point coincides with the boundary between the $q_\infty = 0$ and $q_\infty > 0$ regimes.

Since we are interested in the near-TCP properties of the GEP, we focus on the behavior of $f(q_l)$ for $q_l \ll 1$, which is described by the Taylor expansion

$$f(q_l) = \frac{\zeta_{\alpha-2} - \zeta_{\alpha-1}}{\zeta_{\alpha-1}} \lambda q_l$$

$$+ \frac{\zeta_{\alpha-3} - 3\zeta_{\alpha-2} + 2\zeta_{\alpha-1}}{2\zeta_{\alpha-1}} g_2(\lambda, \mu) q_l^2$$

$$+ \frac{\zeta_{\alpha-4} - 6\zeta_{\alpha-3} + 11\zeta_{\alpha-2} - 6\zeta_{\alpha-1}}{6\zeta_{\alpha-1}} g_3(\lambda, \mu) q_l^3$$

$$+ \frac{\Gamma(2-\alpha)}{\zeta_{\alpha-1}} g_{\alpha-2}(\lambda, \mu) q_l^{\alpha-2} + O\left( q_l^{\min[4,\alpha-1]} \right). \quad (5)$$

Here $\Gamma$ is the Gamma function and

$$g_{\alpha}(\lambda, \mu) \equiv \frac{\mu - \lambda - (1-\lambda)\mu^\alpha}{1-\mu}. \quad (6)$$
the dimensionless parameters $\epsilon$ Here from infection attempts involving $s$

Note that $q$ has a noninteger value greater than $3$, where the solid (dashed) line indicates the continuous (discontinuous) transition line. The dotted lines represent the crossover scales given by $|\epsilon_\mu| \sim |\epsilon_\lambda|^\phi$.

### Table 1. Scaling exponents of GEP on random scale-free networks of degree distribution $P(k) \sim k^{-\alpha}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta_c$</th>
<th>$\beta_\tau$</th>
<th>$\tau$</th>
<th>$d_l$</th>
<th>$\phi$</th>
<th>$\mu_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &gt; 5$</td>
<td>$\frac{5}{2}$</td>
<td>$\frac{7}{2}$</td>
<td>$\frac{5}{2}$</td>
<td>$\frac{d_l}{2}$</td>
<td>$\frac{\phi}{2}$</td>
<td>$\mu_c$</td>
</tr>
<tr>
<td>$4 &lt; \alpha &lt; 5$</td>
<td>$1$</td>
<td>$2$</td>
<td>$\frac{d_l}{2}$</td>
<td>$\frac{\phi}{2}$</td>
<td>$\mu_c$</td>
<td></td>
</tr>
<tr>
<td>$3 &lt; \alpha &lt; 4$</td>
<td>$1$</td>
<td>$2$</td>
<td>$\frac{d_l}{2}$</td>
<td>$\frac{\phi}{2}$</td>
<td>$\mu_c$</td>
<td></td>
</tr>
<tr>
<td>$2 &lt; \alpha &lt; 3$</td>
<td>$\frac{d_l}{2}$</td>
<td>$\frac{\phi}{2}$</td>
<td>$\mu_c$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Note that $q_l^\alpha$ with integer $s$ represents the contribution from infection attempts involving $s$ neighbors. On the other hand, $q_l^-2$ corresponds to contributions from hubs, whose effects become weaker with decreasing $\alpha$. It is clear that $\mu$ contributes only to infection attempts involving more than 2 neighbors.

### III. Numerical Results

**TCP Identification** — We focus on the case when $\alpha$ has a noninteger value greater than 3. For a given value of $\alpha$ satisfying this condition, we can always identify the dominant components of $f(q_l)$. This, together with the approximation $dq_l \approx q_{l+1} - q_l$ for Eqs. (2) and (5), gives

$$dq_l \approx \epsilon_\lambda q_l + u_\alpha \epsilon_\mu q_l^{1+1/\beta_\tau} - v_\alpha q_l^{1+1/\beta_\tau}. \quad (7)$$

Here $u_\alpha$ and $v_\alpha$ are $\alpha$-dependent positive coefficients, the dimensionless parameters $\epsilon_\lambda \equiv (\lambda - \lambda_c)/\lambda_c$ and $\epsilon_\mu \equiv (\mu - \mu_t)/\mu_t$ are defined in terms of

$$\lambda_c = \frac{\zeta_{\alpha-1}}{\zeta_{\alpha-2} - \zeta_{\alpha-1}} = \frac{\langle k \rangle}{\langle k(k-1) \rangle} \quad (8)$$

and a nonzero solution $\mu_t$ of

$$g_2(\lambda_c, \mu_t) = 0 \quad (9)$$

The $\alpha$-dependence of $\mu_t$ is illustrated in Fig. 1, with the switching at $\alpha = 4$ manifest as a cusp singularity. Finally, $\beta_\tau$ and $\beta_\mu$ are $\alpha$-dependent exponents, whose values are listed in Table 1. A phase transition occurs when the fixed-point equation $dq_l = 0$ has an unstable fixed point at $q_l = 0$ and a stable fixed point at $q_l > 0$, so that a positive $q_l$ in the initial state can saturate to $q_\infty > 0$ as $l$ is increased. For $\epsilon_\lambda < 0$, Eq. (7) is approximated as $dq_l \approx -|\epsilon_\lambda|q_l$, which implies an exponential decay $q_l \sim \exp(-|\epsilon_\lambda|l)$ to $q_\infty = 0$. Thus the contagion remains localized to a $\lambda$-dependent length scale $l_\lambda \sim |\epsilon_\lambda|^{-1}$. In contrast, if $\epsilon_\lambda > 0$, the contagion initially spreads as $q_l \sim \exp(\epsilon_\lambda l)$ and eventually saturates to a positive value

$$q_\infty \approx \begin{cases} \left[\epsilon_\lambda/(u_\alpha \epsilon_\mu)\right]^{\beta_\tau} & \text{if } \epsilon_\mu < 0, \\ \epsilon_\lambda/v_\alpha^{\beta_\tau} & \text{if } \epsilon_\mu = 0, \\ \left[u_\alpha \epsilon_\mu/v_\alpha\right]^{\beta_\tau} & \text{if } \epsilon_\mu > 0. \end{cases} \quad (10)$$

This implies that, if $\lambda_c$ obtained from Eq. (8) satisfies $0 < \lambda_c < 1$ (this is true if $k_m$ is sufficiently large), the transition point (epidemic threshold) is given by $\lambda = \lambda_c$. Using Eq. (10) as well as Eq. (4), we observe that for $\epsilon_\mu < 0$ the transition is continuous, with $r \sim \epsilon_\lambda^{\beta_\tau}$ in its vicinity. The critical exponent $\beta_\tau$ is equal to that of the bond percolation (or the ordinary SIR model) on scale-free networks [17]. In contrast, for $\epsilon_\mu > 0$ a discontinuous transition occurs. The marginal case $\epsilon_\mu = 0$ corresponds to a TCP, with $r \sim \epsilon_\lambda^{\beta_\mu}$ in its vicinity.

In Table 1, the tricritical exponent $\beta_\mu$ is always smaller than $\beta_c$, reflecting more rapid increase of $r$ from zero aided by $\mu$. For a given value of $\alpha > 3$, the phase diagram is illustrated in the inset of Fig. 1.

**Divergent length scales.** — There are the two length scales which diverge at $\epsilon_\mu = 0$. We note that Eq. (7) remains invariant under the rescaling $l \rightarrow bl$ if the other variables rescale as

$$q_l \rightarrow b^{-\beta_c}q_{bl}, \quad \epsilon_\lambda \rightarrow b^{-1}\epsilon_\lambda, \quad \epsilon_\mu \rightarrow b^{-\phi}\epsilon_\mu. \quad (11)$$

![Critical Phenomena of the Generalized Epidemic Process – Meesoon HA](image-url)
where the crossover scaling exponent $\phi \equiv 1 - \beta_t / \beta_c$. The rescaling behavior of $\epsilon_\lambda$ reflects the diverging length scale $l_\lambda \sim |\epsilon_\lambda|^{-1}$. Similarly, the rescaling behavior of $\epsilon_\mu$ implies another divergent length scale $l_\mu \sim |\epsilon_\mu|^{-1/\phi}$.

Near TCPs ($\epsilon_\mu = 0$), the spreading is governed by whichever of $l_\lambda$ and $l_\mu$ is larger. For $l_\lambda \gg l_\mu$, the R-cluster exhibits fractal properties of ordinary bond percolation. For $l_\lambda \ll l_\mu$, the R-cluster shows properties which are fractal but still distinct from ordinary bond percolation. For $l_\lambda \ll l_\mu$, the spreading is governed by $\epsilon_\mu$ as $P_r \sim N^{1+\beta_t / \nu_t}$ at the tricritical point. Thus, even at the tricritical point, $P_r$ is governed by the critical exponent $\beta_t$ as $P_r \sim N^{-\beta_t / \nu_t}$.

The hyperscaling relation is as follows:

$$\nu_t = \beta_t + \frac{\beta_c}{\tau - 2},$$

from which we derive $\nu_t$ shown in Table 1.

Near a TCP, the interplay of $l_\lambda$, $l_\mu$, and $l_N$ gives rise to an extended finite-size scaling (FSS) form

$$q_t = l_N^{-\beta_t} \Phi_q(l/l_N, l_\lambda / l_N, l_\mu / l_N).$$

In an actual simulation, the layer index $l$ can be replaced with the time $t$, and instead of $q_t$ it is easier to measure the mean infected cluster size $\langle R \rangle = \int_0^N R P_R dR$, which implies $\langle R \rangle \sim N^{\gamma_t / \nu_t}$ with

$$\gamma_t = 1 - \frac{\beta_t + \beta_c}{\nu_t}.$$
The insets of Fig. 2 show the FSS forms for $\epsilon_\mu = 0$.

In Fig. 3, we also numerically test the extended FSS theory of the time evolution of $\langle R \rangle$ as plotting below and above in the thermodynamic limit where $N \to \infty$. In particular to the FSS form of the characteristic time scales $t_1$ (the correlation time of the ordinary percolation) and $t_2$ (that of TCPs). How to indicate them is shown in the left panel of Fig. 3. For the cases of $\alpha = 5.5, 4.5,$ and 3.5, in the middle and right panels of Fig. 3, the extended FSS theory of $t_2$ is checked as data collapsing for the tricritical exponent $\nu_t$ (middle) and the critical one $\nu_c$ (bottom), respectively.

IV. Summary and discussion

We proposed the extended finite-size scaling (FSS) theory of the generalized epidemic process (GEP) on uncorrelated random scale-free networks with $p_k \sim k^{-\alpha}$ and numerically confirmed associated scaling exponents against $\alpha$ as well as crossover scalings. Numerical studies for $2 < \alpha < 3$ needs to some additional study related to cutoff-degree scalings because such cases are very sensitive to degree-degree correlations and finite-size effects, which is an important future work.

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REFERENCES


